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CS448J: CASVC 2015 @ Stanford
Exercise Sheet 3: Integrating Ordinary Differential Equations

Exercise 1 *(Foucault Pendulum, 8 Points)*

The dynamics of a Foucault pendulum can be described with the equations of motion

$$\ddot{x}_1 - 2\omega\dot{x}_2 - \left(\omega^2 - \frac{g}{l}\right)x_1 = 0,$$
$$\ddot{x}_2 + 2\omega\dot{x}_1 - \left(\omega^2 - \frac{g}{l}\right)x_2 = 0$$

for the Cartesian coordinates (x_1, x_2). In this context l denotes the fix rod length and g the gravity of the earth. The parameter $\omega = 2\pi/T \sin\varphi$ can be calculated from the day length T and the degree of latitude φ.
Reduce the two second order equations of motion to four first order equations of motion and simulate the dynamics of the Foucault pendulum using the explicit Euler and the classical Runge-Kutta (RK4) method. For each case, visualize the trajectory for $T = 30\,\mathrm{s}$, $\varphi = \pi/2$, $l = 5\,\mathrm{m}$, and $g = 10\,\mathrm{m/s}^2$ with initial conditions $(x_1 = x_2 = 1\,\mathrm{m}, \dot{x}_1 = \dot{x}_2 = 0)$ as well as $(x_1 = x_2 = 0, \dot{x}_1 = \dot{x}_2 = 1\mathrm{m/s})$. Interpret your results.

Exercise 2 *(Verlet Integration, 2 + 2 + 4 + 4 = 12 Points)*

In contrast to the explicit Euler method which is based on a first order forward difference approximation, the so-called Verlet integrator uses the central difference approximation to the acceleration

$$a_n \approx \Delta_t q_n = \frac{\frac{q_{n+1}-q_n}{\Delta t} - \frac{q_n - q_{n-1}}{\Delta t}}{\Delta t} = \frac{q_{n+1} - 2q_n + q_{n-1}}{\Delta t^2}$$

leading to the numerical scheme

$$q_{n+1} := 2q_n - q_{n-1} + a_n\Delta t^2.$$

In the following, without a loss of generality, we consider the one-dimensional case (it follows similarly for higher dimensions).

1. Show that the aforementioned second-order scheme is equivalent to the first-order formulation

$$\Phi_{\Delta t} : \mathbb{R}^2 \to \mathbb{R}^2, \quad (q_n, v_n) \mapsto (q_{n+1}, v_{n+1}),$$
$$q_{n+1} := q_n + v_n\Delta t + \tfrac{1}{2}a_n\Delta t^2, \quad v_{n+1} := v_n + \tfrac{1}{2}(a_n + a_{n+1})\Delta t$$

   with $v_n := \dot{q}_n = p_n/m$.

2. Prove that the Verlet integrator is time-reversible.
   (Hint: This symmetric behavior can be verified by interchanging $n \leftrightarrow n+1$ and $\Delta t \leftrightarrow -\Delta t$.)

3. Prove that the Verlet integrator is symplectic.
   (Hint: It follows from the definition of symplecticity, that one has to show ${\Phi'}^{\mathsf{T}}_{\Delta t}\mathsf{adiag}(1,-1){\Phi'}_{\Delta t} = \mathsf{adiag}(1,-1)$ in which ${\Phi'}_{\Delta t}$ denotes the Jacobian of $\Phi_{\Delta t}$ and $\mathsf{adiag}(\cdot)$ an anti-diagonal matrix.)

4. Determine the order of the error of the Verlet integrator.

Notes

- The Homework is due by 10:30am on Oct. 20. Written solutions should be handed in before the lecture. Programming assignments must be submitted by email to your tutor David Hyde <dabh@stanford.edu>.
- In case you have any questions about the assignments, please contact your tutor David Hyde <dabh@stanford.edu> or the instructor Prof. Dominik L. Michels <michels@cs.stanford.edu> directly via email.
- Office hours are every Friday, 10-12 in 208/209 Gates CS Bldg. or by appointment.
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