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CS448J: CASVC 2015 @ Stanford
Exercise Sheet 0: Warming-up

Exercise 1 *(Basic Proofs, 0 + 0 + 0 = 0 Points)*

1. Show that the sum of all *odd* number from 1 to $2n-1$ is equal to n^2 for all $n \in \mathbb{N}$.
   (Hint: The proof that the sum of the first n odd integers is equal the square of n was one of the first proofs based on mathematical induction; see Maurolicus' *Arithmeticorum Libri Duo* from 1575.)
2. Show Bernoulli's inequality: for all $-1 \leq x \in \mathbb{R}$ and $0 \leq n \in \mathbb{N}$ holds $(1+x)^n \geq 1 + nx$.
3. Show Weierstrass' product inequality:
   $$\prod_{i=1}^{n}(1-x_i) \geq 1 - \sum_{i=1}^{n} x_i$$
   for $x_i \in [0,1]$ for all $i \in \{1, \ldots, n\}$ and $n \in \mathbb{N}$.

Exercise 2 *(Basic Algebra, 0 + 0 + 0 = 0 Points)*

The special orthogonal group of dimension $n \in \mathbb{N}$ is defined by

$$\mathbf{SO}(n) := \left\{ \boldsymbol{R} \in \mathbb{R}^{n \times n} | \boldsymbol{R}^\mathsf{T} \boldsymbol{R} = \mathbf{1}_n, \det(\boldsymbol{R}) = 1 \right\},$$

in which $\mathbf{1}_n$ denotes the $(n \times n)$-identity matrix.

1. Prove that $G := (\mathbf{SO}(n), \cdot)$ is a group, in which "$\cdot$" denotes the usual matrix multiplication (i.e. show closure, existence of identity and inverse elements, and associativity).
   Is G abelian? (Hint: Proof by cases.)

An element $\boldsymbol{R} = [\boldsymbol{r_1}, \boldsymbol{r_2}, \ldots, \boldsymbol{r_n}] \in \mathbf{SO}(n)$ is called a n-dimensional rotation matrix. Prove the following statements.

2. From $\boldsymbol{R}^\mathsf{T} \boldsymbol{R} = \mathbf{1}_n$ follows $\forall (i,j) \in \{1, 2, \ldots, n\}^2 : \langle \boldsymbol{r_i} | \boldsymbol{r_j} \rangle = \delta_{ij}$.
3. Each $\boldsymbol{R} \in \mathbf{SO}(3)$ is right-handed oriented.
   (Hint: $\det([\boldsymbol{r_1}, \boldsymbol{r_2}, \boldsymbol{r_3}]) = \langle \boldsymbol{r_1} | \boldsymbol{r_2} \times \boldsymbol{r_3} \rangle$.)

Exercise 3 *(Basic Geometry, 0 Points)*

A mapping $\boldsymbol{g} : \mathbb{R}^3 \to \mathbb{R}^3$ is a rigid body transformation, if and only if

$$\|\boldsymbol{g}(\boldsymbol{p_1}) - \boldsymbol{g}(\boldsymbol{p_2})\| = \|\boldsymbol{p_1} - \boldsymbol{p_2}\|,$$

holds for all $\boldsymbol{p_1}, \boldsymbol{p_2} \in \mathbb{R}^3$ and

$$\boldsymbol{g_*}(\boldsymbol{v_1} \times \boldsymbol{v_2}) = \boldsymbol{g_*}(\boldsymbol{v_1}) \times \boldsymbol{g_*}(\boldsymbol{v_2}) \tag{1}$$

holds for all $\boldsymbol{v_1}, \boldsymbol{v_2} \in \mathbb{R}^3$. In this definition g induces an action on vectors in a natural way by $\boldsymbol{g_*}(\boldsymbol{v_i}) = \boldsymbol{g}(\boldsymbol{q_1}) - \boldsymbol{g}(\boldsymbol{q_2})$ for $\boldsymbol{v_i} = \boldsymbol{q_1} - \boldsymbol{q_2}$.

Show that rigid body transformations always map orthogonal frames to orthogonal frames.
(Hint: Interprete Eq. 1 graphically and show that the inner product $\langle \cdot | \cdot \rangle$ is preserved by rigid body transformations. For that make use of the identity $\langle \boldsymbol{v_1} | \boldsymbol{v_2} \rangle = \frac{1}{4} \left(\|\boldsymbol{v_1} + \boldsymbol{v_2}\|^2 - \|\boldsymbol{v_1} - \boldsymbol{v_2}\|^2 \right)$.)

Exercise 4 *(Basic Calculus, 0 + 0 = 0 Points)*

1. Show that the derivative of the sine is given by the cosine using the definitions

$$\sin(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}, \qquad \cos(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}.$$

2. Verify, that the sinusoidal function given by $x(t) = \cos(\omega_0 t)x(0)$ is a solution of the harmonic oscillator described by the second order ordinary differential equation $\ddot{x} + \omega_0^2 x = 0$.

Exercise 5 *(Analysis of Algorithms, 0 + 0 = 0 Points)*

For an abstract data type *RummageTable* there exists a method *search*(), which can only be applied if *RummageTable* is not empty. This method returns an object from *RummageTable* by sampling from it equally distributed. Let $n \in \mathbb{N}$ and a *RummageTable RT* is initialized with objects labeled with consecutive numbers $1, \ldots, n$. Consider the following loop with an empty body: while W.*suche*() $\neq 1$ do.

1. Determine the probability that the loop terminates?

2. Determine the average runtime in a closed-form expression.

Notes

- This exercise sheet is not graded (0 points).
- For submissions and in case you have any questions about the assignments, please contact your tutor David Hyde <dabh@stanford.edu> or the instructor Prof. Dominik L. Michels <michels@cs.stanford.edu> directly via email.
- Office hours are every Friday, 10-12 in 208/209 Gates CS Bldg. or by appointment.
- The university expects both faculty and students to respect and follow Stanford's Honor Code; see https://communitystandards.stanford.edu/.