The multiresolution Analysis originally developed in [Mallat 1989] and [Meyer 1992] lays the theoretical foundation of the fast wavelet transform (FWT).

Consider a signal $f$ from a subspace $V_{-1}$ of $L^2(\mathbb{R})$, which we would like to decompose in its high frequency (rough) and its low frequency (smooth) part. The smooth part is described by an orthogonal projection $P_0 f$ onto a smaller space $V_0$ containing the smooth functions from $V_{-1}$. The orthogonal complement $W_0$ of $V_0$ in $V_{-1}$ contains the rough parts in $V_{-1}$. Let $P_0$ denote the orthogonal projection onto $W_0$, such that

$$f = P_0 f + Q_0 f, \quad V_{-1} = V_0 \oplus W_0.$$ 

Similarly, $V_0$ is described as the orthogonal sum of $V_1$ and $W_1$. Let $P_1$ and $Q_1$ be the corresponding projections, such that

$$P_0 f = P_1 P_0 f + Q_1 P_0 f, \quad V_0 = V_1 \oplus W_1.$$
Because of $P_1 P_0 f = P_1 f$ and $Q_1 P_0 f = Q_1 f$ we obtain $P_0 f = P_1 f + Q_1 f$ and therefore

$$f = P_1 f + Q_1 f + Q_0 f.$$ 

In the next step, $P_1 f$ is decomposed in $P_2 f$ and $Q_2 f$. Continuing this recursively leads to

$$L^2(\mathbb{R}) \rightarrow V_{-1} \xrightarrow{P_0} V_0 \xrightarrow{P_1} V_1 \rightarrow \{0\}$$

$$\downarrow Q_0 \quad \downarrow Q_1$$

$$\vdots \quad W_0 \quad W_1 \quad \vdots$$

In general, the projections $P_m f$ and $Q_m f$ describe the smooth and rough parts on the scale described by the space $V_{m-1}$.
More precisely, a multiresolution analysis (MRA) of $L^2(\mathbb{R})$ contains a sequence $(V_m)_{m \in \mathbb{Z}}$ of closed subspaces $V_m \subset L^2(\mathbb{R})$ and a scaling function $\phi \in V_0$ with the following properties:

1. $\{0\} \subset \cdots \subset V_2 \subset V_1 \subset V_0 \subset V_{-1} \subset V_{-2} \subset \cdots \subset L^2(\mathbb{R})$,
2. $\bigcup_{m \in \mathbb{Z}} V_m = L^2(\mathbb{R})$ and $\bigcap_{m \in \mathbb{Z}} V_m = \{0\}$,
3. the spaces $V_m$ are scaled versions of $V_0$, i.e. $f(\cdot) \in V_m$ iff $f(2^m \cdot) \in V_0$,
4. the set of the shifts $\phi(\cdot - k)$, $k \in \mathbb{Z}$ is an orthonormal basis of $V_0$.

From (3) and (4) follows that $(\phi_{m,k})_{k \in \mathbb{Z}}$ with $\phi_{m,k}(x) := 2^{-m/2} \phi(2^{-m}x - k)$ is a Hilbert basis of $V_m$. 
Filter Coefficients

For a MRA \( ((V_m)_{m \in \mathbb{Z}}, \varphi) \), the sequence \( h \in l^2(\mathbb{Z}) \) with \( h_k = \langle \varphi | \varphi_{-1,k} \rangle \) fulfills the scaling equation

\[
\varphi = \sum_{k \in \mathbb{Z}} h_k \varphi_{-1,k}
\]

respectively \( \varphi(x) = \sqrt{2} \sum_{k \in \mathbb{Z}} h_k \varphi(2x - k) \).

Furthermore the coefficients \( h_k \) appear on every scale, i.e.

\[
\varphi_{m,k} = \sum_{j \in \mathbb{Z}} h_j \varphi_{m-1,2k+j},
\]

and fulfill the orthogonality relation

\[
\sum_{k \in \mathbb{Z}} h_{k+2j} \bar{h}_k = \delta_{0,j}.
\]
Let us consider the Haar MRA with

\[ \varphi = \chi_{[0,1)} = 2^{-1/2}(\varphi_{-1,0} + \varphi_{-1,1}), \]

\[ h_0 = h_1 = 2^{-1/2} \text{ and } h_k = 0 \text{ for all } k \in \mathbb{Z} \setminus \{0,1\}, \]

and so-called wavelet coefficients \( g \) with \( g_k := (-1)^k \bar{h}_{1-k} \), i.e. \( g_0 = -g_1 = 2^{-1/2} \) and \( g_k = 0 \) for all \( k \in \mathbb{Z} \setminus \{0,1\} \).

The corresponding frequency responses are given by

\[ H(\omega) = 2^{-1/2}(1 + \exp(-2\pi i \omega)), \quad G(\omega) = H(\omega + 1/2). \]

More precisely, it can be shown, that the convolution with \( h \) is a low-pass filter and the convolution with \( g \) is a high pass filter.
Filter Coefficients and Wavelets

The coefficients of $h$ allow for the construction of a mother wavelet $\psi$, whose shifted and scaled versions are Hilbert bases of the spaces $W_m$. More precisely for a MRA $((V_m)_{m \in \mathbb{Z}}, \varphi)$ with scaling coefficients $h \in \ell^1(\mathbb{Z})$ and wavelet coefficients $g_k := (-1)^k h_{1-k}$ we define $\psi \in V_{-1}$ with

$$\psi(x) := \sum_{k \in \mathbb{Z}} g_k \varphi_{-1,k}(x).$$

For the functions

$$\psi_{m,k}(x) := 2^{-m/2} \psi(2^{-m}x - k), \quad m, k \in \mathbb{Z},$$

the following statements hold:

1. $\psi_{m,k} = \sum_{j \in \mathbb{Z}} g_j \varphi_{m-1,2k+j}$,
2. $(\psi_{m,k})_{k \in \mathbb{Z}}$ is a Hilbert basis of $W_m$,
3. $(\psi_{m,k})_{m,k \in \mathbb{Z}}$ is a Hilbert basis of $L^2(\mathbb{R}) = \bigoplus_{m \in \mathbb{Z}} W_m$,
4. $\psi = \psi_{0,0}$ is a wavelet.
Instead of computing the wavelet coefficients $\langle f|\psi^{s,t}\rangle$ by approximating the integral, a discrete version of the signal $f$ is low-pass filtered with $h$ and high-pass filtered with $g$ in the fast wavelet transform (FWT).

Let $((V_m)_{m \in \mathbb{Z}}, \varphi)$ be a MRA with scaling coefficients $\varphi = \sum_{k \in \mathbb{Z}} h_k \varphi^{-1,k}$. Consider $f \in V_0$. Since $(\varphi_{0,k})_{k \in \mathbb{Z}}$ is a Hilbert basis of $V_0$, there exists a uniquely determined sequence $(v^0_k)_{k \in \mathbb{Z}} \in \ell^2(\mathbb{Z})$ with Fourier series representation $f = \sum_{k \in \mathbb{Z}} v^0_k \varphi_{0,k}$. Let $\psi$ be the wavelet corresponding to $\varphi$, i.e. $\psi = \sum_{k \in \mathbb{Z}} g_k \varphi^{-1,k}$ with $g_k := (-1)^k \bar{h}_{1-k}$. Then $(\psi_{m,k})_{m,k \in \mathbb{Z}}$ with

$$\psi_{m,k}(x) = 2^{-m/2} \psi(2^{-m} x - k)$$

is a Hilbert basis of $L^2(\mathbb{R})$ for which reason we only have to evaluate the CTWT $\tilde{f}(s, t)$ at the specific positions $(s, t) \in \{(2^m, 2^m k)|m, k \in \mathbb{Z}\}$. Because of

$$V_0 = \bigoplus_{m > 0} W_m,$$

is it sufficient to compute $\tilde{f}(2^m, 2^m k) = \langle f|\psi_{m,k}\rangle$ for all $m \in \mathbb{Z}_{>0}$ and $k \in \mathbb{Z}$. 
Fast Wavelet Transform (FWT)

We define $\left( w^m_k \right)_{k \in \mathbb{Z}} \in \ell^2(\mathbb{Z})$ with $w^m_k := \langle f | \psi_{m,k} \rangle$ and $\left( v^m_k \right)_{k \in \mathbb{Z}} \in \ell^2(\mathbb{Z})$ with $v^m_k := \langle f | \varphi_{m,k} \rangle$, so that we obtain from $\psi_{m,k} = \sum_{l \in \mathbb{Z}} g_l \varphi_{m-1,2k+l}$ and $\varphi_{m,k} = \sum_{l \in \mathbb{Z}} h_l \varphi_{m-1,2k+l}$, 

$$w^m_k = \sum_{l \in \mathbb{Z}} \bar{g}_{l-2k} v^{m-1}_l, \quad v^m_k = \sum_{l \in \mathbb{Z}} \bar{h}_{l-2k} v^{m-1}_l$$

due to the substitution $l \leftarrow 2k + l$. According to this, we define the operators 

$$\left( H v \right)_k := \sum_{l \in \mathbb{Z}} \bar{h}_{l-2k} v_l, \quad \left( G v \right)_k := \sum_{l \in \mathbb{Z}} \bar{g}_{l-2k} v_l.$$ 

These operators can be seen as convolutions with filter coefficients $\tilde{h}$ respectively $\tilde{g}$ with $\tilde{h}_k = \bar{h}_{-k}$, $\tilde{g}_k = \bar{g}_{-k}$ for $k \in \mathbb{Z}$, followed by a downsampling operator $(\downarrow 2)(x)(n) := x(2n)$, i.e. 

$$w^m = (\downarrow 2)(\tilde{g} * v^{m-1}), \quad v^m = (\downarrow 2)(\tilde{h} * v^{m-1})$$.
Fast Wavelet Transform (FWT)

Fast wavelet transform for input Fourier coefficients $v^0 = (v^0_k)_{k \in \mathbb{Z}}$.

function $\text{FWT}(v^0, M($number of scales$))$

begin
for $m \leftarrow 1$ to $M$ do

$w^m \leftarrow Gv^{m-1}$
$v^m \leftarrow Hv^{m-1}$

end

return $(w^1, \ldots, w^M, v^M)$

end

The mapping $v^0 \mapsto (w^1, \ldots, w^M, v^M)$ given by the FWT is based on the decomposition

$V_0 = \bigoplus_{m=1}^{M} W_m \oplus V_M.$

In practice, samples $f(k) = \langle f | \delta_{0,k} \rangle \approx \langle f | \varphi_{0,k} \rangle = v^0_k$ are used.
Inverse Fast Wavelet Transform (IFWT)

The mapping $v^0 \mapsto (w^1, \ldots, w^M, v^M)$ is an isomorphism. Its inverse can be described using the adjunct operators

$$(\mathcal{H}^* v)_k := \sum_{l \in \mathbb{Z}} h_{k-2l} v_l, \quad (\mathcal{G}^* v)_k := \sum_{l \in \mathbb{Z}} g_{k-2l} v_l,$$

describing an upsampling step with $(\uparrow 2)(x)(n) := x(n/2)$ for even inputs $x$ and $(\uparrow 2)(x)(n) := 0$ for odd inputs $x$, followed by convolutions with filter coefficients $h$ respectively $g$, so that a reconstruction step

$$v^{m-1} = \mathcal{H}^* v^m + \mathcal{G}^* w^m$$

from scale $m$ to scale $m-1$ by can be realized by

$$v^{m-1} = h * (\uparrow 2)(v^m) + g * (\uparrow 2)(w^m).$$
Inverse Fast Wavelet Transform (IFWT)

Inverse Fast wavelet transform for input decomposition \((w^1, \ldots, w^M, v_M)\).

function \(\text{FWT}(w^1, \ldots, w^M, v^M)\)

begin

\[\text{for } m \leftarrow M \text{ down to } 1 \text{ do}\]
\[v^{m-1} \leftarrow H^* v^m + G^* w^m\]

end

return \(v^0\)

end

The complexities of both transform, FWT and IFWT, are in \(\mathcal{O}(|h|N)\), in which \(N\) denotes the length of the input signal and \(|h|\) the length of the filter coefficients \(h\) and \(g\). Since \(|h| \ll N\), it can be considered as a program constant leading to linear complexity \(\mathcal{O}(N)\).