Concepts and Algorithms of Scientific and Visual Computing

–Multiresolution Analysis–

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Multiresolution Analysis

The multiresolution Analysis originally developed in [Mallat 1989] and [Meyer 1992] lays the theoretical foundation of the fast wavelet transform (FWT).

Consider a signal f from a subspace V_{-1} of $L^2(\mathbb{R})$, which we would like to decompose in its high frequency (rough) and its low frequency (smooth) part. The smooth part is described by an orthogonal projection $P_0 f$ onto a smaller space V_0 containing the smooth functions from V_{-1}. The orthogonal complement W_0 of V_0 in V_{-1} contains the rough parts in V_{-1}. Let P_0 denote the orthogonal projection onto W_0, such that

$$f = P_0 f + Q_0 f, \quad V_{-1} = V_0 \oplus W_0.$$

Similarly, V_0 is described as the orthogonal sum of V_1 and W_1. Let P_1 and Q_1 be the corresponding projections, such that

$$P_0 f = P_1 P_0 f + Q_1 P_0 f, \quad V_0 = V_1 \oplus W_1.$$

Multiresolution Analysis

Because of $P_1 P_0 f = P_1 f$ and $Q_1 P_0 f = Q_1 f$ we obtain $P_0 f = P_1 f + Q_1 f$ and therefore

$$f = P_1 f + Q_1 f + Q_0 f.$$

In the next step, $P_1 f$ is decomposed in $P_2 f$ and $Q_2 f$. Continuing this recursively leads to

$$\begin{tikzcd}
L^2(\mathbb{R}) \cdots \arrow[r] & V_{-1} \arrow[r, "P_0"] \arrow[dr, "Q_0"] & V_0 \arrow[r, "P_1"] \arrow[dr, "Q_1"] & V_1 \arrow[r] & \cdots \arrow[r] & \{0\} \\
& \dots & W_0 & W_1 & \dots &
\end{tikzcd}$$

In general, the projections $P_m f$ and $Q_m f$ describe the smooth and rough parts on the scale described by the space V_{m-1}.

Multiresolution Analysis

More precisely, a multiresolution analysis (MRA) of $L^2(\mathbb{R})$ contains a sequence $(V_m)_{m\in\mathbb{Z}}$ of closed subspaces $V_m \subset L^2(\mathbb{R})$ and a scaling function $\varphi \in V_0$ with the following properties:

1. $\{0\} \subset \cdots \subset V_2 \subset V_1 \subset V_0 \subset V_{-1} \subset V_{-2} \subset \cdots \subset L^2(\mathbb{R})$,
2. $\overline{\cup_{m\in\mathbb{Z}} V_m} = L^2(\mathbb{R})$ and $\cap_{m\in\mathbb{Z}} V_m = \{0\}$,
3. the spaces V_m are scaled versions of V_0, i.e. $f(\cdot) \in V_m$ iff $f(2^m\cdot) \in V_0$,
4. the set of the shifts $\varphi(\cdot - k)$, $k \in \mathbb{Z}$ is an orthonormal basis of V_0.

From (3) and (4) follows that $(\varphi_{m,k})_{k\in\mathbb{Z}}$ with $\varphi_{m,k}(x) := 2^{-m/2}\varphi(2^{-m}x - k)$ is a Hilbert basis of V_m.

Filter Coefficients

For a MRA $((V_m)_{m\in\mathbb{Z}}, \varphi)$, the sequence $h \in \ell^2(\mathbb{Z})$ with $h_k = \langle \varphi | \varphi_{-1,k} \rangle$ fulfills the scaling equation

$$\varphi = \sum_{k\in\mathbb{Z}} h_k \varphi_{-1,k}$$

respectively $\varphi(x) = \sqrt{2} \sum_{k\in\mathbb{Z}} h_k \varphi(2x-k)$.

Furthermore the coefficients h_k appear on every scale, i.e.

$$\varphi_{m,k} = \sum_{j\in\mathbb{Z}} h_j \varphi_{m-1,2k+j},$$

and fulfill the orthogonality relation

$$\sum_{k\in\mathbb{Z}} h_{k+2j} \bar{h}_k = \delta_{0,j}.$$

Filter Coefficients

Let us consider the Haar MRA with

$$\varphi = \chi_{[0,1)} = 2^{-1/2}(\varphi_{-1,0} + \varphi_{-1,1}),$$

$h_0 = h_1 = 2^{-1/2}$ and $h_k = 0$ for all $k \in \mathbb{Z} \setminus \{0,1\}$, and so-called wavelet coefficients g with $g_k := (-1)^k \bar{h}_{1-k}$, i.e. $g_0 = -g_1 = 2^{-1/2}$ and $g_k = 0$ for all $k \in \mathbb{Z} \setminus \{0,1\}$.

The corresponding frequency responses are given by

$$H(\omega) = 2^{-1/2}(1 + \exp(-2\pi i\omega)), \quad G(\omega) = H(\omega + 1/2).$$

More precisely, it can be shown, that the convolution with h is a low-pass filter and the convolution with g is a high pass filter.

Filter Coefficients and Wavelets

The coefficients of h allow for the construction of a mother wavelet ψ, whose shifted and scaled versions are Hilbert bases of the spaces W_m. More precisely for a MRA $((V_m)_{m\in\mathbb{Z}}, \varphi)$ with scaling coefficients $h \in \ell^1(\mathbb{Z})$ and wavelet coefficients $g_k := (-1)^k \bar{h}_{1-k}$ we define $\psi \in V_{-1}$ with

$$\psi(x) := \sum_{k\in\mathbb{Z}} g_k \varphi_{-1,k}(x).$$

For the functions

$$\psi_{m,k}(x) := 2^{-m/2}\psi(2^{-m}x - k), \quad m, k \in \mathbb{Z},$$

the following statements hold:

1. $\psi_{m,k} = \sum_{j\in\mathbb{Z}} g_j \varphi_{m-1,2k+j}$,
2. $(\psi_{m,k})_{k\in\mathbb{Z}}$ is a Hilbert basis of W_m,
3. $(\psi_{m,k})_{m,k\in\mathbb{Z}}$ is a Hilbert basis of $L^2(\mathbb{R}) = \overline{\bigoplus_{m\in\mathbb{Z}} W_m}$,
4. $\psi = \psi_{0,0}$ is a wavelet.

Fast Wavelet Transform (FWT)

Instead of computing the wavelet coefficients $\langle f|\psi^{s,t}\rangle$ by approximating the integral, a discrete version of the signal f is low-pass filtered with h and high-pass filtered with g in the fast wavelet transform (FWT).

Let $((V_m)_{m\in\mathbb{Z}},\varphi)$ be a MRA with scaling coefficients $\varphi=\sum_{k\in\mathbb{Z}}h_k\varphi_{-1,k}$. Consider $f\in V_0$. Since $(\varphi_{0,k})_{k\in\mathbb{Z}}$ is a Hilbert basis of V_0, there exists a uniquely determined sequence $(v_k^0)_{k\in\mathbb{Z}}\in\ell^2(\mathbb{Z})$ with Fourier series representation $f=\sum_{k\in\mathbb{Z}}v_k^0\varphi_{0,k}$. Let ψ be the wavelet corresponding to φ, i.e. $\psi=\sum_{k\in\mathbb{Z}}g_k\varphi_{-1,k}$ with $g_k:=(-1)^k\bar{h}_{1-k}$. Then $(\psi_{m,k})_{m,k\in\mathbb{Z}}$ with

$$\psi_{m,k}(x)=2^{-m/2}\psi(2^{-m}x-k)$$

is a Hilbert basis of $L^2(\mathbb{R})$ for which reason we only have to evaluate the CTWT $\tilde{f}(s,t)$ at the specific positions $(s,t)\in\{(2^m,2^mk)|m,k\in\mathbb{Z}\}$. Because of

$$V_0=\overline{\bigoplus_{m>0}W_m},$$

is it sufficient to compute $\tilde{f}(2^m,2^mk)=\langle f|\psi_{m,k}\rangle$ for all $m\in\mathbb{Z}_{>0}$ and $k\in\mathbb{Z}$.

Fast Wavelet Transform (FWT)

We define $(w_k^m)_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ with $w_k^m := \langle f|\psi_{m,k}\rangle$ and $(v_k^m)_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ with $v_k^m := \langle f|\varphi_{m,k}\rangle$, so that we obtain from $\psi_{m,k} = \sum_{l\in\mathbb{Z}} g_l \varphi_{m-1,2k+l}$ and $\varphi_{m,k} = \sum_{l\in\mathbb{Z}} h_l \varphi_{m-1,2k+l}$,

$$w_k^m = \sum_{l\in\mathbb{Z}} \bar{g}_{l-2k} v_l^{m-1}, \quad v_k^m = \sum_{l\in\mathbb{Z}} \bar{h}_{l-2k} v_l^{m-1}$$

due to the substitution $l \leftarrow 2k + l$. According to this, we define the operators

$$(\mathcal{H}v)_k := \sum_{l\in\mathbb{Z}} \bar{h}_{l-2k} v_l, \quad (\mathcal{G}v)_k := \sum_{l\in\mathbb{Z}} \bar{g}_{l-2k} v_l.$$

These operators can be seen as convolutions with filter coefficients $\tilde{h}$ respectively $\tilde{g}$ with $\tilde{h}_k = \bar{h}_{-k}$, $\tilde{g}_k = \bar{g}_{-k}$ for $k \in \mathbb{Z}$, followed by a downsampling operator $(\downarrow 2)(x)(n) := x(2n)$, i.e.

$$w^m = (\downarrow 2)(\tilde{g} * v^{m-1}), \quad v^m = (\downarrow 2)(\tilde{h} * v^{m-1}).$$

Fast Wavelet Transform (FWT)

Fast wavelet transform for input Fourier coefficients $v^0 = (v_k^0)_{k\in\mathbb{Z}}$.

```
function FWT(v^0, M(number of scales))
begin
    for m ← 1 to M do
        w^m ← G v^{m-1}
        v^m ← H v^{m-1}
    end
    return (w^1, ..., w^M, v^M)
end
```

The mapping $v^0 \mapsto (w^1, \ldots, w^M, v^M)$ given by the FWT is based on the decomposition

$$V_0 = \left(\bigoplus_{m=1}^{M} W_m \right) \oplus V_M.$$

In practice, samples $f(k) = \langle f|\delta_{0,k}\rangle \approx \langle f|\varphi_{0,k}\rangle = v_k^0$ are used.

Inverse Fast Wavelet Transform (IFWT)

The mapping $v^0 \mapsto (w^1, \ldots, w^M, v^M)$ is an isomorphism. Its inverse can be described using the adjunct operators

$$(\mathcal{H}^* v)_k := \sum_{l \in \mathbb{Z}} h_{k-2l} v_l, \quad (\mathcal{G}^* v)_k := \sum_{l \in \mathbb{Z}} g_{k-2l} v_l,$$

describing an upsampling step with $(\uparrow 2)(x)(n) := x(n/2)$ for even inputs x and $(\uparrow 2)(x)(n) := 0$ for odd inputs x, followed by convolutions with filter coefficients h respectively g, so that a reconstruction step

$$v^{m-1} = \mathcal{H}^* v^m + \mathcal{G}^* w^m$$

from scale m to scale $m-1$ by can be realized by

$$v^{m-1} = h * (\uparrow 2)(v^m) + g * (\uparrow 2)(w^m).$$

Inverse Fast Wavelet Transform (IFWT)

Inverse Fast wavelet transform for input decomposition $(w^1, \ldots, w^M, v_M)$.

```
function FWT(w^1,...,w^M, v^M))
begin
    for m ← M down to 1 do
        v^{m-1} ← H* v^m + G* w^m
    end
    return v^0
end
```

The complexities of both transform, FWT and IFWT, are in $\mathcal{O}(|h|N)$, in which N denotes the length of the input signal and $|h|$ the length of the filter coefficients h and g. Since $|h| \ll N$, it can be considered as a program constant leading to

linear complexity $\mathcal{O}(N)$.