Concepts and Algorithms of Scientific and Visual Computing

–Discrete Fourier Transforms–

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Discrete-time Fourier Transform (DTFT)

As in the continuous case, for $x \in \ell^2(\mathbb{Z})$, the discrete-time Fourier transform (DTFT) can be defined by

$$\hat{x}(\omega) := \sum_{k \in \mathbb{Z}} x(k) \exp(-2\pi i \omega k)$$

for all $\omega \in [0, 1]$.

The mapping

$$\ell^2(\mathbb{Z}) \ni x \mapsto \hat{x} \in L^2([0, 1])$$

defines an unitary automorphism with the inverse transform given by

$$x(k) = \int_0^1 \hat{x}(\omega) \exp(2\pi i \omega k) \, \mathrm{d}\omega.$$

Discrete-time Fourier Transform (DTFT)

Let $f \in L^2(\mathbb{R})$ be a piecewise continuous function and $x \in \ell^2(\mathbb{Z})$ the sampled function of f with sampling rate one, i.e. $x(k) = f(k)$ for all $k \in \mathbb{Z}$. According to the definitions of the CTFT and the DTFT, we obtain

$$\hat{f}(\omega) = \int_{\mathbb{R}} f(t) \exp(-2\pi i \omega t)\, \mathrm{d}t, \quad \hat{x}(\omega) = \sum_{k \in \mathbb{Z}} f(k) \exp(-2\pi i \omega k),$$

so that $\hat{x}(\omega)$ is a Riemann sum of $\hat{f}(\omega)$ for all $\omega \in \mathbb{R}$.

Please note, that the sample functions are not able to detect oscillations with frequencies larger one, which can lead to artifacts in the Riemann approximation. This effect is well-known as aliasing.

In praxis the closely related discrete cosine transformation (DCT), see [Ahmed 1974], is often used in the context of the compression of audio, image, and video data, e.g. in the JPEG compression algorithm or in its modified version (MDCT) in the MP3 compression algorithm.

Windowed Fourier Transform (WFT)

So far, the Fourier analysis of a signal comes with a significant disadvantage: since the exponential function $t \mapsto \exp(2\pi i \omega t)$ is periodic, it can not be localized with respect to time and is therefore not appropriate for the analysis of aperiodic signals. More precisely, the frequency information can be seen as a mean value with respect to the whole time interval and the temporal information is hidden in the phase. To overcome this shortcoming, Dennis Gábor introduced the windowed Fourier transform in 1946, which is a compromise between the time- and the frequency-based representation of a signal.

Windowed Fourier Transform (WFT)

To obtain the temporal information, it only considers a small sector of the signal $f \in L^2(\mathbb{R})$ for the spectral analysis. This is realized with the use of an appropriate window function $g \in L^2(\mathbb{R})$ with $\|g\| \neq 0$ centered around zero, so that the translation $u \mapsto g(u-t)$ is centered around t. To determine the distribution of the frequencies of f around t, f is pointwise multiplied with the t translation of g and the resulting product is then analyzed.

For a given window function $g \in L^2(\mathbb{R})$, we define (musical) notes of frequency ω at time t with

$$g_{\omega,t}(u) := g(u-t)\exp(2\pi i \omega u)$$

for $u \in \mathbb{R}$.

Please not, that $\left\|g_{\omega,t}\right\| = \|g\|$ implies that all notes $g_{\omega,t}$ are elements of $L^2(\mathbb{R})$.

Windowed Fourier Transform (WFT)

Let $g \in L^2(\mathbb{R})$ be a window function. For $f \in L^2(\mathbb{R})$, the mapping $\tilde{f} : \mathbb{R}^2 \to \mathbb{C}$ defined by

$$\tilde{f}(\omega, t) := \langle f | g_{\omega,t} \rangle = \int_{\mathbb{R}} f(u) \bar{g}(u - t) \exp(-2\pi i \omega u) \, \mathrm{d}u$$

is called the windowed Fourier transform (WFT) of f with respect to g.

The mapping

$$u \mapsto \langle f | g_{\omega,t} \rangle g_{\omega,t}(u)$$

can be seen as a projection of the signal f onto the note $g_{\omega,t}$.

Heisenberg's Uncertainty Principle

The window function g should be chosen in a way, that g localizes optimally with respect to time and $\hat{g}$ localizes optimally with respect to frequency. Unfortunately, it turned out that this can not be realized, since there is a natural limitation called Heisenberg's uncertainty principle.

Originally introduced in quantum mechanics, the uncertainty principle, formulated by Werner K. Heisenberg in [Heisenberg 1927], states a fundamental limit to the precision with which certain pairs of physical properties of a particle (e.g. the position x and the momentum p) can be known simultaneously. More precisely, the lower bound is given by

$$\sigma_x \sigma_p \geq \frac{\hbar}{2},$$

in which σ_x and σ_p denote the standard deviations of x and p, and $\hbar := h/2\pi$ denotes the reduced Planck constant.

Heisenberg's Uncertainty Principle

Let us briefly define some basic stochastic properties of the window function $g \in L^2(\mathbb{R})$.

Center of g:

$$t_0 = t_0(g) := \int_{\mathbb{R}} t\,|g(t)|^2 \,\mathrm{d}t,$$

width of g:

$$T = T(g) := \left(\int_{\mathbb{R}} (t - t_0)^2 \,|g(t)|^2 \,\mathrm{d}t \right)^{1/2},$$

center of $\hat{g}$:

$$\omega_0 = \omega_0(g) := \int_{\mathbb{R}} \omega\,|\hat{g}(\omega)|^2 \,\mathrm{d}\omega,$$

width of $\hat{g}$:

$$\Omega = \Omega(g) := \left(\int_{\mathbb{R}} (\omega - \omega_0)^2 \,|\hat{g}(\omega)|^2 \,\mathrm{d}\omega \right)^{1/2}.$$

Center and width are expected value and standard deviation of $t \mapsto |g(t)|^2$.

Heisenberg's Uncertainty Principle

For each $g \in L^2(\mathbb{R})$ with $|g| = 1$ Heisenberg's uncertainty principle

$$T(g) \cdot \Omega(g) \geq \frac{1}{4\pi}$$

holds.

The lower bond is exactly taken by the Gaussian function

$$g_{\omega_0, t_0}(t) := \pi^{-1/4} \frac{1}{\sqrt{2\pi}} \exp(-2\pi i \omega_0 t) \exp(-\pi (t - t_0)^2)$$

which follows easily by calculation. Furthermore, the Gaussian is the only function with minimal uncertainty. The WFT with $g := g_{\omega_0, t_0}$ is known as Gábor transform, see [Gábor 1946].

Heisenberg's Uncertainty Principle

We provide a brief sketch of the proof of Heisenberg's uncertainty principle here.

W.l.o.g. we assume that

$$g \in \left\{ g \in L^2(\mathbb{R}) \,|\, g \text{ continuously differentiable and } g' \in L^2(\mathbb{R}) \right\} \subset L^2(\mathbb{R}),$$

as well as $T(g) < \infty$ and $\Omega(g) < \infty$, g and $\hat{g}$ are centered, i.e. $t_0 := t_0(g) = 0$ and $\omega_0 := \omega_0(g) = 0$.

Because of $\|h\| = 1$ and

$$\hat{h}(\omega) = \exp(2\pi i t_0(\omega + \omega_0)) g(\omega + \omega_0),$$

h and $\hat{h}$ are also centered.

Heisenberg's Uncertainty Principle

Furthermore

$$\int_{\mathbb{R}} t^2 |h(t)|^2 \, \mathrm{d}t \cdot \int_{\mathbb{R}} \omega^2 \left|\hat{h}(t)\right|^2 \mathrm{d}\omega = \int_{\mathbb{R}} (t - t_0)^2 |g(t)|^2 \, \mathrm{d}t \cdot \int_{\mathbb{R}} (\omega - \omega_0)^2 |\hat{g}(t)|^2 \, \mathrm{d}\omega.$$

Using $\omega \hat{g}(\omega) = \hat{g}'(\omega)/(2\pi i)$ and the Parseval equality $\left\|\hat{g}'\right\| = \|g'\|$, we get

$$T(g)^2 \cdot \Omega(g)^2 = \int_{\mathbb{R}} t^2 |g(t)|^2 \, \mathrm{d}t \cdot \int_{\mathbb{R}} \omega^2 |\hat{g}(t)|^2 \, \mathrm{d}\omega = \frac{1}{4\pi^2} \int_{\mathbb{R}} t^2 |g(t)|^2 \, \mathrm{d}t \cdot \int_{\mathbb{R}} \left|g'(t)\right|^2 \mathrm{d}t$$

so that we can obtain

$$T(g)^2 \cdot \Omega(g)^2 \geq \frac{1}{4\pi^2} \left(\int_{\mathbb{R}} \left| t g(t) g'(t) \right| \mathrm{d}t \right)^2$$

using the Cauchy-Schwarz inequality $\|f_1\|^2 \|f_2\|^2 \geq |\langle f_1 | f_2 \rangle|^2$ with $f_1(t) = |tg(t)|$ and $f_2(t) = |g'(t)|$.

Heisenberg's Uncertainty Principle

For arbitrary $a, b \in \mathbb{C}$ it holds $|ab| = \left|a\bar{b}\right| \geq \mathrm{Re}(a\bar{b}) = (a\bar{b} + \bar{a}b)/2$. With $a := tg(t)$ and $b := g'(t)$ we obtain

$$T(g)^2 \cdot \Omega(g)^2 \geq \frac{1}{4\pi^2}\left(\int_{\mathbb{R}} \left|tg(t)g'(t)\right| \mathrm{d}t\right)^2 \geq \frac{1}{4\pi^2}\left(\frac{1}{2}\int_{\mathbb{R}} (tg(t)\bar{g'}(t) + t\bar{g}(t)g'(t))\,\mathrm{d}t\right)^2.$$

Using $\mathrm{d}_t |g(t)|^2 = g(t)\bar{g'}(t) + g'(t)\bar{g}(t)$, $\int_{\mathbb{R}} t|g(t)|^2 \,\mathrm{d}t = t_0(g) = 0$, and $\lim_{t\to\infty} t|g(t)|^2 = 0$, integration by parts leads to

$$\frac{1}{4\pi^2}\left(\frac{1}{2}\int_{\mathbb{R}} (tg(t)\bar{g'}(t) + t\bar{g}(t)g'(t))\,\mathrm{d}t\right)^2 \geq \frac{1}{16\pi^2}\left(-\int_{\mathbb{R}} |g(t)|^2 \,\mathrm{d}t\right)^2 = \frac{1}{16\pi^2}\|g\|^4.$$

The uncertainty principle follows from $\|g\| = 1$.