Concepts and Algorithms of Scientific and Visual Computing

–Signal Analysis–

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Review: Fourier Transform

For functions $f \in L^1(\mathbb{R}^n)$, the Fourier transform of f is given by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \xi \cdot x} dx$$

and the corresponding inverse Fourier transform of $\hat{f}$ is given by

$$f(x) = \int_{\mathbb{R}^n} \hat{f}(\xi) e^{2\pi i \xi \cdot x} d\xi$$

where x, ξ are vectors in $\mathbb{R}^n$.
Recall: $f \in L^1(\mathbb{R}^n) \iff$

$$\int_{\mathbb{R}^n} |f(x)| \, dx < \infty$$

This matters—signal vanishes at extreme frequencies (Riemann-Lebesgue Lemma).

Riemann-Lebesgue Lemma

Lemma (Riemann-Lebesgue): If f is L^1 integrable on $\mathbb{R}^n$, then

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \xi \cdot x} dx \to 0 \text{ as } |\xi| \to \infty.$$

Proof (n=1): Suppose f is smooth and has compact support. First apply integration by parts to $|\hat{f}|$, yielding

$$\left| \int_{\mathbb{R}} f(x) e^{-2\pi i \xi x} dx \right| = \left| \int_{\mathbb{R}} \frac{1}{2\pi i \xi} f'(x) e^{-2\pi i \xi x} dx \right|$$

$$\leq \frac{1}{|\xi|} \int_{\mathbb{R}} |f'(x)| dx \to 0 \text{ as } |\xi| \to \infty$$

So $|\hat{f}| \to 0 \implies \hat{f} \to 0$ as $\xi \to \pm\infty$. (continued on next slide)

Riemann-Lebesgue Lemma

If f does not have compact support, then since it is L^1, it can be approximated arbitrarily closely by a smooth function g that <u>does</u> have compact support. So pick a g such that $\|f-g\|_{L^1} < \epsilon$. Then

$$|\hat{f}| = \left| \int_{\mathbb{R}} (f-g)e^{-2\pi i \xi x} dx + \int_{\mathbb{R}} g e^{-2\pi i \xi x} dx \right|$$

$$\leq \left| \int_{\mathbb{R}} (f-g)e^{-2\pi i \xi x} dx \right| + \left| \int_{\mathbb{R}} g e^{-2\pi i \xi x} dx \right|$$

The right-hand term goes to zero by the previous slide, and the left-hand term can be bounded by ϵ (L^1 assumption). Therefore, the whole term goes to zero. $\square$

So as long as f is L^1 integrable, its Fourier modes vanish at $\pm\infty$. (Multi-dimensional proof is basically identical)

Fourier Transform Examples

The Fourier transform of a Gaussian $f(x) = e^{-ax^2}$ is also a Gaussian:

$$\mathscr{F}(f) = \int_{-\infty}^{\infty} e^{-ax^2} e^{-2\pi i x \xi} dx = \int_{-\infty}^{\infty} e^{-(ax^2 + 2\pi i \xi x)} dx$$

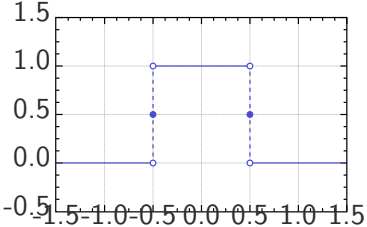
Complete the square inside the expontential. Want $\left(\sqrt{a}x + b\right)^2$ inside the exponent $\implies b = \pi i \xi / \sqrt{a}$. So multiply both sides of equation by $e^{\pi^2 \xi^2 / a}$:

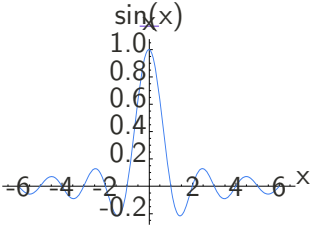
$$e^{\pi^2 \xi^2 / a} \hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-(\sqrt{a}x + b)^2} dx = \int_{-\infty}^{\infty} e^{-z^2} dz = \sqrt{\pi}$$

$$\implies \hat{f}(\xi) = \sqrt{\pi} e^{-\pi^2 \xi^2 / a}$$

Fourier Transform Examples

Not all Fourier transforms are invertible! Consider rectangular function (L^1 integrable), whose Fourier transform the sinc function, which is not Lebesgue integrable. (Image credits: Wikipedia. Display errors: mine.)





Fourier Transform Examples

Demo time... (musical analysis + harmonics)

Real-time Fourier transform visualization

Some Fourier Transform Properties

We can see that differentiation in the time domain corresponds to multiplication in the Fourier domain:

$$\mathscr{F}\left(f'(x)\right)(\xi) = \int_{-\infty}^{\infty} f'(x) e^{-2\pi i \xi x} dx$$

Integration by parts $\implies$

$$= f(x) e^{-2\pi i \xi x}\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} 2\pi i \xi f(x) e^{-2\pi i \xi x} dx$$

$$= 2\pi i \xi \mathscr{F}\left(f(x)\right)(\xi)$$

$\implies$ The Fourier transform can make our lives easier!

Some Fourier Transform Properties

For two functions $f, g \in L^1(\mathbb{R}^n)$, their convolution is defined to be the function

$$h(z) = \int_{\mathbb{R}^n} f(x)g(z-x)dx$$

and the pointwise product is just the function given by $f(x)g(x)$.
The Fourier transform of a convolution is a pointwise product in frequency space:

$$\hat{h}(\xi) = F(\xi)G(\xi)$$

where F and G are the respective Fourier transforms of f and g. Let's derive this...

Some Fourier Transform Properties

First: does h even have a Fourier transform? Check:

$$\|h\|_{L^1} = \int_{\mathbb{R}^n} |h(x)| dx = \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} f(x) g(z-x) dz \right| dx$$

$$\leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x) g(z-x)| \, dz dx = \int_{\mathbb{R}^n} |f(x)| \int_{\mathbb{R}^n} |g(z-x)| dz dx$$

$$= \int_{\mathbb{R}^n} |f(x)| \|g\|_{L^1} dx = \|f\|_{L^1} \|g\|_{L^1} < \infty$$

since $f, g \in L^1(\mathbb{R}^n)$. So $h \in L^1(\mathbb{R}^n)$, which means that h does indeed have a valid Fourier transform. Now let's show that the Fourier transform is what we claim...

Some Fourier Transform Properties

$$H(\xi) = \int_{\mathbb{R}^n} h(z) e^{-2\pi i \xi \cdot z} dz = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x) g(z-x) dx e^{-2\pi i \xi \cdot z} dz$$

$$= \int_{\mathbb{R}^n} f(x) \left(\int_{\mathbb{R}^n} g(z-x) e^{-2\pi i \xi \cdot z} dz \right) dx \text{ by Fubini's Theorem}$$

$$= \int_{\mathbb{R}^n} f(x) \left(\int_{\mathbb{R}^n} g(y) e^{-2\pi i \xi \cdot (y+x)} dy \right) dx \text{ substituting } y = z - x$$

$$= \int_{\mathbb{R}^n} f(x) e^{-2\pi i \xi \cdot x} \left(\int_{\mathbb{R}^n} g(y) e^{-2\pi i \xi \cdot y} dy \right) dx = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \xi \cdot x} dx \int_{\mathbb{R}^n} g(y) e^{-2\pi i \xi \cdot y} dy$$

which is exactly $F(\xi)G(\xi)$. So the Fourier transform of a convolution is a pointwise product. $\square$

Some Fourier Transform Properties

Similar properties hold with the convolution / pointwise product. Convolution Theorem says

$$H(\xi) = F(\xi)G(\xi) \text{ (just proved)}$$

$$\widehat{fg}(\xi) = F(\xi) * G(\xi) \text{ (convolution)}$$

$$(f * g)(x) = \mathscr{F}^{-1}(F(\xi)G(\xi))$$

$$(fg)(x) = \mathscr{F}^{-1}(F(\xi) * G(\xi))$$

Sanity check: why do we care about all these properties we've been deriving?

Solving a Simple Differential Equation with Fourier Transforms

The ODE

$$\frac{d^2y}{dt^2} + 2r\frac{dy}{dt} + \omega_0^2 y = \delta(t - t^*)$$

is the equation of motion for a driven, damped simple harmonic oscillator (think: spring system with friction) with "damping constant" r and natural frequency ω_0, with some impulse instantaneously applied at positive time $t = t^*$.
How do we solve this ODE? Not totally obvious... Fourier transforms to the rescue!

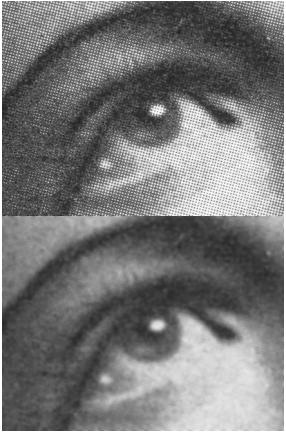
$$\left(-4\pi^2\xi^2 + 4r\pi i\xi + \omega_0^2\right)\hat{y} = e^{-2\pi i\xi t^*}$$

$$\implies y(t) = \int_{-\infty}^{\infty} \frac{e^{-2\pi i\xi t^*}}{-4\pi^2\xi^2 + 4r\pi i\xi + \omega_0^2} e^{2\pi i\xi t} d\xi$$

Still a hard (this one not impossible) integral, but this is a better situation to be in

Some Applications of Convolutions

Convolving an image with a Gaussian function yields a smoothed version of the image (Gaussian Bur) (image credit Wikipedia):



(Will discuss signal filtering like this more on Thursday...)

Some Applications of Convolutions

Convolution reverb: convolve arbitrary audio recording with an impulse response, an audio sample that tells how an (approximately) ideal impulse reverberates in an environment. Result is the input audio recording sounding like it's reverberating in that environment.

Demo

Uncertainty Principle

Compare the time-domain and frequency-domain plots of various Gaussians. What do you see?

Interactive Demo

Uncertainty Principle

The dispersion about zero $D_0(f)$ of a function f is given by the second moment of $|f(x)|$:

$$D_0(f) = \int_{-\infty}^{\infty} x^2 |f(x)|^2 dx$$

(assume f is normalized in the L^2 sense). Under proper smoothness assumptions, the Uncertainty Principle states that

$$D_0(f) D_0\left(\hat{f}\right) \geq \frac{1}{16\pi^2}$$

(Pinsky, Mark (2002), Introduction to Fourier Analysis and Wavelets).
So the more spread out / concentrated f becomes, the more concentrated / spread out $\hat{f}$ becomes.

Uncertainty Principle

What function is "most certain?"
Recall: the Fourier transform of a Gaussian is a Gaussian, with possibly different normalization constants.
For

$$f(x) = \frac{2^{1/4}}{\sqrt{\sigma}} e^{-\pi x^2/\sigma^2},$$

$$\hat{f}(\xi) = \sigma \frac{2^{1/4}}{\sqrt{\sigma}} e^{-\pi \sigma^2 \xi^2}.$$

One can plug into the definition of dispersion about zero to see that for this Fourier transform pair,

$$D_0(f) D_0\left(\hat{f}\right) = \frac{1}{16\pi^2}$$

which means that this function is close as you can get to beating the Uncertainty Principle.
Next time: Heisenberg uncertainty principle in particular