Concepts and Algorithms of Scientific and Visual Computing

–Fourier Analysis–

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L^p Function Spaces

In functional analysis, the Lebesgue spaces L^p (named after Henri L. Lebesgue) are function spaces defined by the canonical extension of the p-norm for finite-dimensional vector spaces.

For $p \in [1, \infty)$, the Lebesgue space $L^p(\mathbb{R})$ contains all measurable functions $f : \mathbb{R} \to \mathbb{C}$ with finite p-norm:

$$L^p(\mathbb{R}) := \left\{ f : \mathbb{R} \to \mathbb{C} \,|\, f \text{ measurable and } \int_{\mathbb{R}} |f(t)|^p \, \mathrm{d}t < \infty \right\}.$$

For $p \to \infty$, we similarly obtain

$$L^\infty(\mathbb{R}) := \left\{ f : \mathbb{R} \to \mathbb{C} \,|\, f \text{ measurable and } \operatorname*{ess\,sup}_{t \in \mathbb{R}} |f(t)| < \infty \right\}.$$

with the essential supremum

$$\operatorname*{ess\,sup}_{t \in \mathbb{R}} |f(t)| := \inf \{ a \geq 0 \,|\, \mu(\{ t \,|\, |f(t)| > a \}) = 0 \},$$

in which μ denotes the Borel measure on $\mathbb{R}$.

L^p Banach Spaces

For $p \in [1, \infty]$, $L^p(\mathbb{R})$ is a linear subspace of $\mathbb{C}^{\mathbb{R}} := \{f : \mathbb{R} \to \mathbb{C}\}$.

We can define norms given by

$$L^p(\mathbb{R}) \ni f \mapsto \|f\|_p := \left(\int_{\mathbb{R}} |f(t)|^p \, \mathrm{d}t \right)^{1/p}$$

for $1 \leq p < \infty$, and

$$L^\infty(\mathbb{R}) \ni f \mapsto \|f\|_\infty := \operatorname{ess\,sup} \{ |f(t)| \mid t \in \mathbb{R} \},$$

so that $L^p(\mathbb{R})$ and $L^\infty(\mathbb{R})$ become complete normed vector (Banach) spaces.

L^2 Hilbert Space

For $p = 2$ we obtain the Hilbert space with the 2-norm

$$\|f\|_2^2 := \langle f | f \rangle$$

induced by the inner product

$$\langle f | g \rangle := \int_{\mathbb{R}} f(t) \bar{g}(t) \, \mathrm{d}t.$$

This is the only Hilbert space of all $L^1(\mathbb{R}), \ldots, L^\infty(\mathbb{R})$ spaces.

ℓ^p Function Space

Similarly to the continuous case, for a given sequence $x : \mathbb{Z} \to \mathbb{C}$ we can define the function space

$$\ell^p(\mathbb{Z}) := \left\{ x : \mathbb{Z} \to \mathbb{C} \,|\, \sum_{n \in \mathbb{Z}} |x_n|^p < \infty \right\}.$$

for $p \in [1, \infty)$ as an analog for discrete domains.

For $p \to \infty$, we similarly obtain

$$\ell^\infty(\mathbb{Z}) := \{ x : \mathbb{Z} \to \mathbb{C} \,|\, \exists B > 0 \forall n \in \mathbb{Z} : |x_n| \leq B \}.$$

As in the continuous case, for $p \in [1, \infty]$, $\ell^p(\mathbb{Z})$ is a linear subspace of $\mathbb{C}^{\mathbb{Z}}$.

ℓ^p Banach Spaces

We can define norms given by

$$\ell^p(\mathbb{Z}) \ni x \mapsto \|x\|_p := \left(\sum_{n \in \mathbb{Z}} |x_n|^p \right)^{1/p}$$

for $1 \leq p < \infty$, and

$$\ell^\infty(\mathbb{Z}) \ni x \mapsto \|x\|_\infty := \sup \{|x_n| \mid n \in \mathbb{Z}\},$$

so that $\ell^p(\mathbb{Z})$ and $\ell^\infty(\mathbb{Z})$ become Banach spaces.

For $p = 2$ we obtain the Hilbert space with the 2-norm

$$\|x\|_2^2 := \langle x | x \rangle$$

induced by the inner product.

$$\langle x | y \rangle := \sum_{n \in \mathbb{Z}} x_n \bar{y}_n.$$

This is the only Hilbert space of all $\ell^1(\mathbb{R}), \dots, \ell^\infty(\mathbb{R})$ spaces.

Fourier Coefficients

Let us restrict $L^2(\mathbb{R})$ to the Hilbert space $L^2([0,1])$ in the canonical sense. A Hilbert basis of $L^2([0,1])$ is given by $\{e_k \,|\, k \in \mathbb{Z}\}$ with $e_k(t) := \exp(2\pi ikt)$ for $t \in [0,1]$, so that a signal $f \in L^2([0,1])$ can be decomposed with

$$f = \sum_{k\in\mathbb{Z}} \langle f|e_k\rangle e_k,$$

in which

$$\langle f|e_k\rangle = \int_0^1 f(t)\exp(-2\pi ikt)\,\mathrm{d}t$$

denotes the so-called Fourier coefficients.

The mapping

$$f \mapsto \hat{f} := (\langle f|e_k\rangle)_{k\in\mathbb{Z}},$$

which returns for a given signal f its series of Fourier coefficients, is an isomorphism $L^2([0,1]) \to \ell^2(\mathbb{Z})$ between Hilbert spaces preserving the inner product and the norm.

Continuous-time Fourier Transform (CTFT)

More general, for $f \in L^2(\mathbb{R})$ the continuous-time Fourier transform (CTFT) can be defined by

$$\omega \mapsto \hat{f}(\omega) := \int_{\mathbb{R}} f(t) \exp(-2\pi i \omega t)\, \mathrm{d}t,$$

measuring the intensity of a given frequency $\omega \in \mathbb{R}$.

It can be shown, that its inverse transform is given by

$$t \mapsto \check{g}(t) := \int_{\mathbb{R}} g(\omega) \exp(2\pi i \omega t)\, \mathrm{d}\omega.$$

According to the Plancherel theorem, the CTFT $f \mapsto \hat{f}$ defines a unitary transform $L^2(\mathbb{R}) \to L^2(\mathbb{R})$ preserving the inner product, the norm, and the energy.