Concepts and Algorithms of Scientific and Visual Computing

–Lie Theory–

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Lie Theory

A well structured general approach to the solution of differential equations has been introduced by M. Sophus Lie in 1870. He showed that the established theories can be summarized using the symmetry properties of the differential equations.

For that, infinitesimal transformations are applied mapping solutions of differential equations to other solutions of differential equations.

Continuous group theory, so-called Lie algebra, and differential geometry are usually used to analyze the underlying structure of the differential equations capturing their deeper connections.

Infinitesimal Symmetry

Let us consider a system of differential equations of order q with n independent and m dependent variables:

$$F_k(\mathbf{x}, \mathbf{u}^{(q)}) = 0, \qquad k \in \{1, \ldots, p\}.$$

A symmetry is a transformation that maps solutions of the system of differential equations to (other) solutions.

The vector field

$$\mathcal{X} = \sum_{i=1}^{n} \xi^i(\mathbf{x}, \mathbf{u}) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{m} \theta_\alpha(\mathbf{x}, \mathbf{u}) \frac{\partial}{\partial u^\alpha},$$

is an infinitesimal symmetry or symmetry generator, if its flow $\exp(a\mathcal{X})$ is a one-parameter symmetry group of the differential system.

Infinitesimal Determining Equations

To find the infinitesimal symmetry, the vector field $\mathcal{X}$ is prolonged to the jet space whose coordinates are the derivatives occurring in the differential system:

$$\mathcal{X}^{(q)} \equiv \mathcal{X}^{(pr)} = \sum_{i=1}^{n} \xi^i \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{m} \sum_{\#J=0}^{q} \theta_\alpha^J \frac{\partial}{\partial u_J^\alpha},$$

where

$$\theta_\alpha^J = D_J \left(\theta^\alpha - \sum_{i=1}^{n} u_i^\alpha \xi^i \right) + \sum_{i=1}^{n} u_{J,i}^\alpha \xi^i,$$

and D_J is the total derivative operator.

Infinitesimal Determining Equations

The infinitesimal invariance criterion reads

$$\mathcal{X}^{(pr)}(F_k) = 0 \qquad \text{whenever} \qquad F_k = 0, \qquad k \in \{1, \ldots, p\}.$$

It generates an overdetermined system

$$\mathcal{L}\left(\mathbf{x}, \mathbf{u}\,; \xi^{(q)}, \theta^{(q)}\right) = 0,$$

of linear partial differential equations in ξ^i and θ^α called infinitesimal determining equations.