Concepts and Algorithms of Scientific and Visual Computing

–Finite Element Method–

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Finite Element Method (FEM)

We consider the partial differential equation

$$\partial_x^2 u(x,y) + \partial_y^2 u(x,y) = f(x,y)$$

with $(x,y) \in G$ and $u(x,y) = 0$ for $(x,y) \in \partial G$. In the following, we develop an exemplary finite element method in order to solve the upper equation.

First, we formulate the corresponding variational problem. For that, we multiply the differential equation with smooth functions $v(x,y)$ which vanish at the boundary, and integrate over G:

$$\int\int_G (\partial_x^2 u + \partial_y^2 u) v \, \mathrm{d}x\mathrm{d}y = \int\int_G f \, v \, \mathrm{d}x\mathrm{d}y.$$

According to Gauß's theorem, this is equivalent to $a(u,v) = b(v)$ with

$$a(u,v) = -\int\int_G (\partial_x u \partial_x v + \partial_y u \partial_y v) \, \mathrm{d}x\mathrm{d}y, \quad b(v) = \int\int_G f \, v \, \mathrm{d}x\mathrm{d}y.$$

Triangulation

We decompose G is small parts using an appropriate triangulation; e.g. for the unit square we simply define sampling points

$$x_\mu = \mu \Delta x, \quad y_\nu = \nu \Delta y$$

with $\mu, \nu \in \{0, \ldots, N\}$ and $\Delta x = \Delta y = 1/N =: h$.

Triangulation

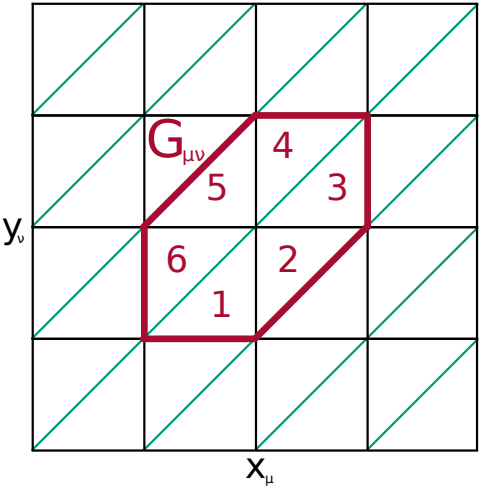

Figure : Illustration of the triangulation of a squared surface.

Basis Functions

For each triangle we formulate a basis function, which is typically of polynomial kind. We employ a simple linear approach

$$u(x,y) \approx \tilde{u}(x,y) = a_1 + a_2 x + a_3 y.$$

The basis functions should provide a continuous transition from one triangle to another, so that the overall solution can be continuous.

The basis coefficients a_1, a_2, a_3 be uniquely determined using the function values u_1, u_2, u_3 at the triangle's corners.

Summation over all triangles leads to

$$\tilde{u}(x,y) = \sum_{\mu=1}^{N-1} \sum_{\nu=1}^{N-1} \alpha_{\mu,\nu} u_{\mu,\nu}(x,y),$$

in which the basis coefficients α must be determined and $u_{\mu,\nu}$ represents a linear function over every triangle.

Basis Functions

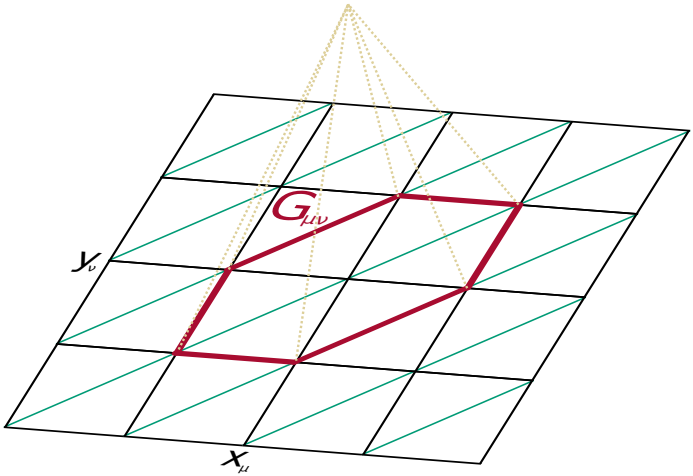

Figure : Illustration of the pyramid-shaped basis functions.

Basis Functions

We set up $u_{\mu,\nu}(x_k, y_l) = 1$ for $k = \mu$ and $l = \nu$, and $u_{\mu,\nu}(x_k, y_l) = 0$ for all other sampling points of $G_{\mu,\nu}$. Moreover, $u_{\mu,\nu}(x, y) = 0$ for $(x, y) \notin G_{\mu,\nu}$.

We illustrate the computation of $u_{\mu,\nu}$ over $G_{\mu,\nu}$ in the case of a triangular of first kind.

From

$$u_{\mu,\nu}(x, y) = a_1 + a_2 x + a_3 y$$

follows $u_{\mu,\nu} = 1$ for $x = x_\mu, y = y_\nu$, $u_{\mu,\nu} = 0$ for $x = x_{\mu-1}, y = y_{\nu-1}$, and $u_{\mu,\nu} = 0$ for $x = x_\mu, y = y_{\nu-1}$, leading to $a_1 = 1 - \nu, a_2 = 0, a_3 = 1/h$.

Hence

$$u_{\mu,\nu}(x, y) = 1 + \left(\frac{y}{h} - \nu\right).$$

Basis Functions

Similarly, we obtain

$$u_{\mu,\nu}(x,y) = 1 - \left(\frac{x}{h} - \mu\right) + \left(\frac{y}{h} - \nu\right), \qquad \text{(Case 2)}$$

$$u_{\mu,\nu}(x,y) = 1 - \left(\frac{x}{h} - \mu\right), \qquad \text{(Case 3)}$$

$$u_{\mu,\nu}(x,y) = 1 - \left(\frac{y}{h} - \nu\right), \qquad \text{(Case 4)}$$

$$u_{\mu,\nu}(x,y) = 1 + \left(\frac{x}{h} - \mu\right) + \left(\frac{y}{h} - \nu\right), \qquad \text{(Case 5)}$$

$$u_{\mu,\nu}(x,y) = 1 + \left(\frac{x}{h} - \mu\right). \qquad \text{(Case 6)}$$

Basis Coefficients

We substitute $\tilde{u}(x,y)$ for $u(x,y)$ and $u_{\mu,\nu}(x,y)$ for $v(x,y)$ into the variational formulation, so that we obtain the linear system

$$\sum_{\mu=1}^{N-1}\sum_{\nu=1}^{N-1} \alpha_{\mu,\nu}\, a(u_{\mu,\nu}, u_{k,l}) = b(u_{k,l})$$

with

$$a(u_{\mu,\nu}, u_{k,l}) = \int\int_{G_{k,l}} (\partial_x u_{\mu,\nu}\, \partial_x u_{k,l} + \partial_y u_{\mu,\nu}\, \partial_y u_{k,l})\, \mathrm{d}x\mathrm{d}y$$

and

$$b(u_{k,l}) = \int\int_{G_{k,l}} f\, u_{k,l}\, \mathrm{d}x\mathrm{d}y.$$

for $k, l \in \{1, \ldots, N-1\}$ to determine the basis coefficients.

Basis Coefficients

Since the integration is performed over simple triangles, the integrals can easily be determined leading to

$$\frac{1}{h^2}(4\alpha_{k,l} - 2\alpha_{k+1,l} - 2\alpha_{k-1,l})\frac{h^2}{2}$$

for the portions caused by the partial derivatives with respect to x, and

$$\frac{1}{h^2}(4\alpha_{k,l} - 2\alpha_{k,l+1} - 2\alpha_{k,l-1})\frac{h^2}{2}$$

for the portions caused by the partial derivatives with respect to y.

Furthermore, we simply obtain $b(u_{k,l}) \approx f_{k,l} V_P$ in which

$$V_P = \frac{1}{3} 6 \frac{1}{2} h^2$$

denotes the volume of the pyramid leading to $b(u_{k,l}) \approx f_{k,l} h^2$.

Solution

Finally, this leads to the linear system

$$4\alpha_{k,l} - \alpha_{k+1,l} - \alpha_{k-1,l} - \alpha_{k,l+1} - \alpha_{k,l-1} = h^2 f_{k,l},$$

$k, l \in \{1, \ldots, N-1\}$, for the basis coefficients.

After computation and substitution of the basis coefficients α,

$$\tilde{u}(x,y) = \sum_{\mu=1}^{N-1} \sum_{\nu=1}^{N-1} \alpha_{\mu,\nu} u_{\mu,\nu}(x,y),$$

is an explicit representation of an approximation of the final solution.

Boundary Element Method (BEM)

In contrast to the finite element method, the boundary element method only discretizes the boundary of a given surface or volume, so that the unknown variables are only located there.

The partial differential equations are transformed to integral equations describing the considered phenomena on the whole domain. These equations are the discretized and solved analogously to the procedure in the finite element method.

The number of notes is usually significantly smaller compared to the finite element and the finite difference methods. Instead, one obtains are usually dense and antisymmetric linear system, which sometimes impedes the solution process.

Furthermore, the boundary element methods can be coupled easily with the finite element method (BEM-FEM coupling).