Consider the ordinary differential equation

\[ d_x(y(x)) = f(x, y(x)). \]

According to the famous theorem of Augustin Cauchy and Sofia V. Kovalevskaya, there exists a solution \( y \) with the initial value \( y_0(x_0) = y_0 \) and continuity in a well defined interval around \( x_0 \), if \( f \) is continuous in the neighborhood \( G \) of the point \( (x_0, y_0) \) defined by \( |x - x_0| < a \) and \( |y - y_0| < b \).

Furthermore, if \( f \) is Lipschitz continuous, i.e.

\[ |f(x, y_1) - f(x, y_2)| \leq N|y_1 - y_2| \]

for all \( (x, y_1) \) and \( (x, y_2) \) in \( G \) and a constant \( N \), then the solution \( y \) is unique according to the theorem named after Émile Picard and Ernst L. Lindelöf.
Numerical Integration

Although, there is a solution to an initial value (Cauchy) problem of ordinary kind according to the Cauchy-Kovalevskaya theorem, it is often not possible to formulate the solution as a composition of analytical expressions.

In such cases, we have to determine particular solutions by applying numerical methods.

Without a loss of generalization, we will focus on the formulation of numerical schemes for ordinary Cauchy problems of first order given by

\[ y' := \frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0. \]

In particular, for the function \( y \), we are searching for appropriate numerical approximations of \( y_i := y(x_i) \) at the sampling points \( x_i \).
(Explicit) Euler Method

The integration of the Cauchy problem leads to

$$y(x) = y_0 + \int_{x_0}^{x} f(x, y(x)) \, dx.$$ 

For $x_1 := x_0 + \Delta x$ we obtain

$$y(x_1) = y_0 + \int_{x_0}^{x_0 + \Delta x} f(x, y(x)) \, dx \approx y_0 + \Delta x f(x_0, y_0) =: y_1.$$ 

For equidistant sampling points $x_i := x_0 + i\Delta x$, this can be generalized to the so-called (explicit) Euler method

$$y_{i+1} = y_i + \Delta x f(x_i, y_i).$$
According to Taylor’s theorem,

\[ y(x_1) = y(x_0 + \Delta x) = y_0 + f(x_0, y_0)\Delta x + \frac{y''(x_0)}{2}\Delta x^2 + \ldots \]

holds, so that the explicit Euler method is of first order corresponding to an error

\[ |y(x_1) - y_1| \in O(\Delta x^2) \]

of second order.
Classical Runge-Kutta Method (RK4, 4th-order)

function RK4
begin
\[\begin{aligned}
    &x_i \leftarrow i \Delta x \\
    &y_i' \leftarrow F(x_i, y_i) \\
    &y_A \leftarrow y_i + \frac{\Delta x}{2} y_i' \\
    &y_A' \leftarrow F(x_i + \frac{\Delta x}{2}, y_A) \\
    &y_B \leftarrow y_i + \frac{\Delta x}{2} y_A' \\
    &y_B' \leftarrow F(x_i + \frac{\Delta x}{2}, y_B) \\
    &y_C \leftarrow y_i + \Delta x y_B' \\
    &y_C' \leftarrow F(x_i + \Delta x, y_C) \\
    &y_{i+1} \leftarrow y_i + \frac{\Delta x}{6} \left( y_i' + 2(y_A' + y_B') + y_C' \right)
\end{aligned}\]
end
return \((y_1, y_2, \ldots, y_N)\)
Linear Multistep Methods

The (explicit) Euler and the classical Runge-Kutta method are so-called (linear) one-step methods since the computation of $y_{i+1}$ only requires $y_i$.

More general, linear multistep methods are given by

$$y_{i+k} + \alpha_{k-1}y_{i+k-1} + \alpha_{k-2}y_{i+k-2} + \cdots + \alpha_1y_{i+1} + \alpha_0y_i = \Delta t(\beta_k f_{i+k} + \beta_{k-1}f_{i+k-1} + \cdots + \beta_1f_{i+1} + \beta_0f_i)$$

with appropriate constants $\alpha_j$ and $\beta_j$. For $\alpha_k \neq 0$ and $\beta_k \neq 0$ such a scheme is called a linear $k$-step method.

It is called explicit, if $\beta_k = 0$ holds, so that only the already known approximation values $y_i, \ldots, y_{i+k-1}$ occur on the right side. For $\beta_k \neq 0$, the method is called implicit because the new approximation value $y_{i+k}$ occurs on both sides.
Explicit vs. Implicit in Practice

Figure: Numerical phase space of a pendulum with one degree of freedom integrated with an explicit, an implicit, and a structure-preserving (covered in the next lecture) numerical method.