Exercise 1 (Multiresolution Analysis)

We obtain \( \psi_{m,k}(x) = \sum_{j \in \mathbb{Z}} g_j \phi_{m-1,2k+j}(x) \), which proves the first statement. Using this result and the relation \( \sum_{k \in \mathbb{Z}} h_{k+2j} \bar{h}_k = \delta_{0,j} \) we get \( \langle \psi_{m,k} | \psi_{n,l} \rangle = \delta_{m,n} \delta_{l,k} \), so that \( (\psi_{m,k})_{m,k \in \mathbb{Z}} \) is an orthonormal system. Furthermore, \( \langle \psi_{0,k} | \phi_{0,l} \rangle = 0 \) implies that all \( \psi_{0,k} \in V_{-1} \) are elements of \( W_0 \). To verify completeness of the orthonormal system, it is sufficient to show Parseval’s identity \( \|x\|^2 = \sum_{b \in B} |\langle x | h \rangle|^2 \) with \( B := \{ \phi_{0,k}, \psi_{0,k} | k \in \mathbb{Z} \} \) for \( x = \varphi_{-1,0} \) which completes the proof of the other statements.

Exercise 2 (Laplace Transform)

We can easily verify the linearity of the Laplace transform and \( \mathcal{L}(\dot{f})(s) = sF(s) - f_0 \), so that we obtain \( F(s) = f_0/(s+\lambda) \) and finally the well known solution \( f(t) = \mathcal{L}^{-1}(f_0/(s+\lambda))(t) = f_0 \exp(-\lambda t) \) of the first-order linear Cauchy problem.