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Autumn 2015

CS448J: CASVC 2015 @ Stanford
Solutions of Exercise Sheet 8: Fast Fourier Transform

Exercise 1 *(Inverse Discrete Fourier Transform)*

For all $k \in \{0, \dots, n-1\}$ we obtain

$$\begin{aligned} p_{\boldsymbol{\varphi}}(\Omega_n^k) &= \sum_{j=0}^{n-1} \varphi_j \Omega_n^{kj} = \sum_{j=0}^{n-1} \frac{1}{n} \left(\sum_{r=0}^{n-1} \zeta_r (\Omega_n^{-j}) r \right) \Omega_n^{kj} \\ &= \sum_{r=0}^{n-1} \zeta_r \cdot \frac{1}{n} \sum_{j=0}^{n-1} \Omega_n^{j(k-r)} = \zeta_k. \end{aligned}$$

Exercise 2 *(Fast Polynomial Multiplication)*

The product of two polynomials can be computed as follows: first, the FFT algorithm is applied to both polynomials which requires time $2 \cdot \mathcal{O}(n \log(n))$; second, the pointwise product of the two results from the first step is computed which requires time $\mathcal{O}(n)$; and third, the inverse FFT is applied to the result from the second step which requires time $\mathcal{O}(n \log(n))$. The leads to a complexity of $\mathcal{O}(n \log(n))$.

Exercise 3 *(FFT Big Theta Time Complexity)*

From

$$\boldsymbol{D}^2 = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & & & 1 \\ \vdots & & \iddots & \\ 0 & 1 & & \end{pmatrix}.$$

we can follow $|\det \boldsymbol{D}|^2 = |\det \boldsymbol{D}^2| = n^n$ leading to at least $c \cdot \log(|\det D|) = c \cdot \log(n^{n/2}) = \cdot \frac{c}{2} n \log(n)$ operations defining a lower bound for the time complexity of the FFT. Furthermore, according to Sheet 7 Exercise 3, the runtime of the FFT is in $\mathcal{O}(n \log(n))$, so that we obtain a time complexity of $\Theta(n \log(n))$ as a combination of both results.

Exercise 4 *(Wavelet Construction)*

With $\hat{\phi}'(\omega) = 2\pi i \omega \hat{\phi}(\omega)$ and $0 \neq \psi := \phi'$ we obtain

$$\begin{aligned} 0 < c_\psi &= \int_{\mathbb{R}} \frac{|\hat{\psi}(\omega)|^2}{|\omega|} \, \mathrm{d}\omega = \int_{|\omega| \leq 1} \frac{|\hat{\psi}(\omega)|^2}{|\omega|} \, \mathrm{d}\omega + \int_{|\omega| > 1} \frac{|\hat{\psi}(\omega)|^2}{|\omega|} \, \mathrm{d}\omega \\ &= \int_{|\omega| \leq 1} \frac{|2\pi\omega\hat{\phi}(\omega)|^2}{|\omega|} \, \mathrm{d}\omega + \int_{|\omega| > 1} \frac{|\hat{\psi}(\omega)|^2}{|\omega|} \, \mathrm{d}\omega \\ &\leq 4\pi^2 \int_{|\omega| \leq 1} |\hat{\phi}(\omega)|^2 \, \mathrm{d}\omega + \int_{|\omega| > 1} |\hat{\psi}(\omega)|^2 \, \mathrm{d}\omega \\ &= 4\pi^2 \|\hat{\phi}\|^2 + \|\hat{\psi}\|^2 = 4\pi \|\phi\|^2 + \|\psi\|^2 < \infty \end{aligned}$$

because $\phi, \psi \in L^2(\mathbb{R})$.