Exercise 1 (Inverse Discrete Fourier Transform)

For all \( k \in \{0, \ldots, n - 1\} \) we obtain

\[
p_\varphi(\Omega_k^n) = \sum_{j=0}^{n-1} \varphi_j \Omega_j^n = \sum_{j=0}^{n-1} \frac{1}{n} \left( \sum_{r=0}^{n-1} \zeta_r (\Omega_j^n)^r \right) \Omega_j^{kn} = \sum_{r=0}^{n-1} \zeta_r \cdot \frac{1}{n} \sum_{j=0}^{n-1} \Omega_j^{(k-r)n} = \zeta_k.
\]

Exercise 2 (Fast Polynomial Multiplication)

The product of two polynomials can be computed as follows: first, the FFT algorithm is applied to both polynomials which requires time \( 2 \cdot O(n \log(n)) \); second, the pointwise product of the two results from the first step is computed which requires time \( O(n) \); and third, the inverse FFT is applied to the result from the second step which requires time \( O(n \log(n)) \). This leads to a complexity of \( \Theta(n \log(n)) \).

Exercise 3 (FFT Big Theta Time Complexity)

From

\[
D^2 = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \downarrow & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 1 \\ \end{pmatrix}
\]

we can follow \(|\det D|^2 = |\det D^2| = n^n\) leading to at least \( c \cdot \log(\det D) = c \cdot \log(n^{n/2}) = \cdot \times n \log(n)\) operations defining a lower bound for the time complexity of the FFT. Furthermore, according to Sheet 7 Exercise 3, the runtime of the FFT is in \( \mathcal{O}(n \log(n)) \), so that we obtain a time complexity of \( \Theta(n \log(n)) \) as a combination of both results.

Exercise 4 (Wavelet Construction)

With \( \hat{\phi}'(\omega) = 2\pi i \omega \hat{\phi}(\omega) \) and \( 0 \neq \psi := \phi' \) we obtain

\[
0 < c_\psi = \int_{\mathbb{R}} \frac{\hat{\psi}(\omega)^2}{|\omega|} d\omega = \int_{|\omega| \leq 1} \frac{\hat{\psi}(\omega)^2}{|\omega|} d\omega + \int_{|\omega| > 1} \frac{\hat{\psi}(\omega)^2}{|\omega|} d\omega \\
= \int_{|\omega| \leq 1} \frac{|2\pi \omega \hat{\phi}(\omega)|^2}{|\omega|} d\omega + \int_{|\omega| > 1} \frac{\hat{\psi}(\omega)^2}{|\omega|} d\omega \\
\leq 4\pi^2 \int_{|\omega| \leq 1} |\hat{\phi}(\omega)|^2 d\omega + \int_{|\omega| > 1} |\hat{\psi}(\omega)|^2 d\omega \\
= 4\pi^2 ||\hat{\phi}||^2 + ||\hat{\psi}||^2 = 4\pi ||\phi||^2 + ||\psi||^2 < \infty
\]

because \( \phi, \psi \in L^2(\mathbb{R}) \).