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High Fidelity Algorithmics Group
CS Dept. Stanford University

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CS448J: CASVC 2015 @ Stanford
Exercise Sheet 8: Fast Fourier Transform

Exercise 1 *(Inverse Discrete Fourier Transform, 4 Points)*

Prove the identity

$$\mathrm{FFT}^{-1}(\boldsymbol{\zeta}, \Omega_n) = \frac{1}{n}\mathrm{FFT}(\boldsymbol{\zeta}, \Omega_n^{-1})$$

for the inverse of the discrete Fourier transform, in which $\boldsymbol{\zeta}$ denotes the result of a Fourier transform of an input signal of length n and $\Omega_n := \exp(2\pi i/n)$ the primitive n-th root of unity.

Exercise 2 *(Fast Polynomial Multiplication, 4 Points)*

Prove that the product of two polynomials of degree n can be computed with time complexity $\mathcal{O}(n\log(n))$.

Exercise 3 *(FFT Big Theta Time Complexity, 7 Points)*

The result $\boldsymbol{\zeta} = (\zeta_0, \ldots, \zeta_{n-1})$ of the FFT for an input signal given as a tuple $\boldsymbol{x} = (x_0, \ldots, x_{n-1})$ of length n can also be obtained by evaluating the matrix-vector product $\boldsymbol{D}\boldsymbol{x} = \boldsymbol{\zeta}$ respectively

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \Omega_n & \Omega_n^2 & \cdots & \Omega_n^{n-1} \\ 1 & \Omega_n^2 & \Omega_n^4 & \cdots & \Omega_n^{2(n-1)} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & \Omega_n^{n-1} & \Omega_n^{2(n-1)} & \cdots & \Omega_n^{(n-1)(n-1)} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{n-1} \end{pmatrix} = \begin{pmatrix} \zeta_0 \\ \zeta_1 \\ \zeta_2 \\ \vdots \\ \zeta_{n-1} \end{pmatrix}.$$

According to Morgenstern's theorem[1], the evaluation of this product requires at least $c \cdot \log(|\det D|)$ operations with constant $c \in \mathbb{R}$. Estimate the time complexity of the FFT in Big Theta notation. (Hint: Calculate the determinant of $\boldsymbol{D}^2$ first.)

Exercise 4 *(Wavelet Construction, 7 Points)*

Prove that if $\phi \in L^2(\mathbb{R})$ is differentiable with $0 \neq \psi := \phi' \in L^2(\mathbb{R})$, then ψ is a wavelet.
(Hint: $\hat{\phi}'(\omega) = 2\pi i\omega\hat{\phi}(\omega)$.)

[1] See Jacques Morgenstern, "Note on a lower bound on the linear complexity of the fast Fourier transform', 'J. ACM, 20(2):305–306, 1973.

Notes

- The Homework is due by 10:30am on Nov. 24. Written solutions should be handed in before the lecture. Programming assignments must be submitted by email to your tutor David Hyde <dabh@stanford.edu>.
- In case you have any questions about the assignments, please contact your tutor David Hyde <dabh@stanford.edu> or the instructor Prof. Dominik L. Michels <michels@cs.stanford.edu> directly via email.
- Office hours are every Friday, 10-12 in 208/209 Gates CS Bldg. or by appointment.
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