Exercise 1 (Fermi-Pasta-Ulam Problem, 10 Points)

Simulate the dynamics of a simple Fermi-Pasta-Ulam problem which is described by the equations of motion
\[
\ddot{x} + \frac{1}{m} \begin{bmatrix} k_a + k_b & -k_b \\ -k_b & k_a + k_b \end{bmatrix} x = \frac{1}{m} \begin{bmatrix} k_a \dot{x}_0 \cos(\omega t) \\ 0 \end{bmatrix},
\]
in which \(k_b\) denotes the stiffness of the middle spring and \(k_a\) denotes the stiffness of both other springs. At initial time the system resides in an equilibrium state and is then excited by an external force acting sinusoidally with a frequency of \(\omega\) and an amplitude of \(\dot{x}_0\).

The ratio \(\kappa\) of the extremal eigenvalues of the coefficient matrix is a measure of the stiffness of the system. Because of \(\kappa = 1 + 2k_b/k_a\) the stiffness of the system is large for \(k_b \gg k_a\).

Plot the trajectories \(x_1(t)\) and the phase portraits \(x_2(x_1)\), and \(v_1(x_1)\) of this Fermi-Pasta-Ulam problem for its solution computed with the Gautschi type exponential time integrator and the variational implicit-explicit integrator. Make use of the parameter setting \(m = 1.0, k_a = 10^2, k_b = 10^{10}, \dot{x}_0 = 1.0,\) and \(\omega = \sqrt{k_a/m - 10^{-10}}\).

Interpret your results using a comparison with the phase portraits determined by the analytical solution which is given by \(\dot{x}(t) = \dot{x}_0 \cos(\omega t)\) with the amplitude
\[
\dot{x} := \frac{k_a}{m} \dot{x}_0 \left(\omega_1^2 - \omega^2\right)^{-1} \left(\omega_2^2 - \omega^2\right)^{-1} \left[\frac{\omega_3^2 - \omega^2}{\omega_4^2 - \omega_1^2}\right],
\]
and the natural frequencies of \(\omega_1^2 := k_a/m, \omega_2^2 := (k_a + 2k_b)/m,\) and \(\omega_3^2 := (k_a + k_b)/m\).

Notes

- The Homework is due by 10:30am on Oct. 27. Written solutions should be handed in before the lecture. Programming assignments must be submitted by email to your tutor David Hyde <dabh@stanford.edu>.
- In case you have any questions about the assignments, please contact your tutor David Hyde <dabh@stanford.edu> or the instructor Prof. Dominik L. Michels <michels@cs.stanford.edu> directly via email.
- Office hours are every Friday, 10-12 in 208/209 Gates CS Bldg. or by appointment.
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