FALL 2016, CS448J:CASVC EXERCISE SHEET 2

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1. Beltrami Identity

First we recall the standard Euler-Lagrangian equation $\partial_q L = d_t(\partial_{\dot{q}} L)$. With an additional $\partial_t L = 0$ given to us, we aim to reformulate the equation. Given this knowledge, we know that

$$d_t L = (\partial_q L)\dot{q} + (\partial_{\dot{q}} L)\ddot{q} + (\partial_t L) = (\partial_q L)\dot{q} + (\partial_{\dot{q}} L)\ddot{q}.$$
$$(\partial_q L)\dot{q} = d_t L - (\partial_{\dot{q}} L)\ddot{q}$$

By multiplying both sides of the Euler-Lagrangian equation by $\dot{q}$, moving all terms to the left side, then subsituting in the above equality, we get

$$(\partial_q L)\dot{q} - d_t(\partial_{\dot{q}} L)\dot{q} = (d_t L - (\partial_{\dot{q}} L)\ddot{q}) - d_t(\partial_{\dot{q}} L)\dot{q} = 0$$

Using the rearranged product rule $d_t(uv) - u(d_t v) = (d_t u)v$ with $u = \partial_{\dot{q}} L$ and $v = \dot{q}$, we can further simplify this equality by

$$\begin{aligned}(d_t L - (\partial_{\dot{q}} L)\ddot{q}) - d_t(\partial_{\dot{q}} L)\dot{q} &= d_t L - (\partial_{\dot{q}} L)\ddot{q} - (d_t((\partial_{\dot{q}} L)\dot{q}) - (\partial_{\dot{q}} L)\ddot{q}) \\ &= d_t L - d_t((\partial_{\dot{q}} L)\dot{q}) \\ &= d_t(L - (\partial_{\dot{q}} L)\dot{q}) = 0\end{aligned}$$

Therefore, by integrating both sides by t, we obtain

$$L - (\partial_{\dot{q}} L)\dot{q} = c \text{ for some constant } c.$$

2. Shortest Path

We can take advantage of the proof of the Euler Lagrangian already shown in class, where the extremal $\int_{t_i}^{t_f} L dt$ is obtained when $\partial_q L = d_t(\partial_{\dot{q}} L)$, and generalize it to find the shortest path. Given a path $\alpha : [t_i, t_f] \to \mathbb{R}^2$, the length of the path is given by

$$\int_{t_i}^{t_f} \sqrt{1 + (\dot{\alpha})^2} dt$$

And the extremal (minimal) value is obtained when $\partial_\alpha L = d_t(\partial_{\dot{\alpha}} L)$ with $L(\alpha, \dot{\alpha}, t) = \sqrt{1 + (\dot{\alpha})^2}$. Expanding the terms gives us

$$0 = \partial_\alpha L = d_t(\partial_{\dot{\alpha}} L) = d_t(\frac{\dot{\alpha}}{\sqrt{1 + (\dot{\alpha})^2}})$$

And integrating both sides by t gives us

$$\begin{aligned} c &= \frac{\dot{\alpha}}{\sqrt{1 + (\dot{\alpha})^2}} \\ c^2(1 + \dot{\alpha})^2 &= \dot{\alpha}^2 \\ c^2 &= (1 - c^2)\dot{\alpha}^2 \end{aligned}$$

$$\sqrt{\frac{c^2}{1 - c^2}} = \dot{\alpha} \text{ for some constant } c \in [-1, 1]$$

Note that when we take the square root in the last step, we do not consider the case of the negative sign, as from the first equation we know that $\dot{\alpha}$ and c are of the same sign.Therefore, the shortest path will be the curve with a constant tangent, which is a straight line.

3. Soap Film

The area of the surface generated by rotating $(x, y(x))$ along the x axis is given by

$$A(y, \dot{y}, t) = \int y\sqrt{1 + (\dot{y})^2}\ dx$$

And we wish to find the minimal surface, where this area is minimized. We observe that A is not explicitly dependent on time, and thus using the simplified Euler-Lagrangian as shown in exercise 1 with $L = A$, we obtain

$$d_t(L - (\partial_{\dot{q}} L)\dot{q}) = d_t(y\sqrt{1 + (\dot{y})^2} - \frac{y \cdot (\dot{y})^2}{\sqrt{1 + (\dot{y})^2}}) = d_t(\frac{y + y(\dot{y})^2 - y(\dot{y})^2}{\sqrt{1 + (\dot{y})^2}}) = d_t(\frac{y}{\sqrt{1 + (\dot{y})^2}}) = 0$$

$$\Rightarrow \frac{y}{\sqrt{1 + (\dot{y})^2}} = c_1 \Rightarrow y^2 = (c_1)^2(1 + (\dot{y})^2) \Rightarrow \pm\frac{\sqrt{y^2 - (c_1)^2}}{c_1} = \dot{y}$$

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From this we can obtain two system of solutions in the form of a separable first order ordinary differential equation,

$$\frac{1}{\sqrt{y^2-(c_1)^2}}\dot{y} = \frac{1}{c_1} \text{ and } \frac{-1}{\sqrt{y^2-(c_1)^2}}\dot{y} = \frac{1}{c_1}$$

By rewriting this as $\frac{1}{\sqrt{y^2-(c_1)^2}}dy = \pm\frac{1}{c_1}dx$, we can integrate both sides and get

$$\begin{aligned}
&\log(\sqrt{y^2-(c_1)^2}+y)+c_2 = \pm\frac{x}{c_1}\\
\Rightarrow &\log c_2(\sqrt{y^2-(c_1)^2}+y)) = \pm\frac{x}{c_1}\\
\Rightarrow &c_2(\sqrt{y^2-(c_1)^2}+y) = e^{\pm\frac{x}{c_1}}\\
\Rightarrow &\sqrt{y^2-(c_1)^2} = c_2e^{\pm\frac{x}{c_1}}-y\\
\Rightarrow &y^2-(c_1)^2 = (c_2)^2e^{\pm 2\frac{x}{c_1}}-2e^{\pm\frac{x}{c_1}}y+y^2\\
\Rightarrow &2e^{\pm\frac{x}{c_1}}y = (c_1)^2+(c_2)^2e^{\pm 2\frac{x}{c_1}}\\
\Rightarrow &y = \frac{1}{2}((c_1)^2e^{\mp\frac{x}{c_1}}+(c_2)^2e^{\pm\frac{x}{c_1}})
\end{aligned}$$

(Note that c_2 here simply represents some constant, so for brevity, we slightly abuse notation and change the value c_2 corresponds to throughout our calculations.)So for the soap film to be a minimal surface, $y(x)$ has to satisfy

$$y = \frac{1}{2}((c_1)^2e^{-\frac{x}{c_1}}+(c_2)^2e^{\frac{x}{c_1}}) \text{ or } y = \frac{1}{2}((c_1)^2e^{\frac{x}{c_1}}+(c_2)^2e^{-\frac{x}{c_1}})$$

4. Higher Order Euler-Lagrangian

Using notation from the derivation of the Euler-Lagrangian, dependent up to the first derivative, we see that

$$d_\epsilon\int_{t_i}^{t_f} L(\tilde{q},\dot{\tilde{q}},\dots,\tilde{q}^{(n)},t)dt = \int_{t_i}^{t_f}(\partial_{\tilde{q}}L)\alpha\ dt + \sum_{i=1}^{n}\int_{t_i}^{t_f}(\partial_{\tilde{q}^{(i)}}L)\alpha^{(i)}dt$$

For each partial derivative of order k, we can perform partial integration k times, getting

$$\begin{aligned}
\int_{t_i}^{t_f}(\partial_{\tilde{q}^{(k)}}L)\alpha^{(k)}dt &= (\partial_{\tilde{q}^{(k)}}L)\alpha^{(k-1)}|_{t_i}^{t_f} - \int_{t_i}^{t_f} d_t(\partial_{\tilde{q}^{(k)}}L)\alpha^{(k-1)}dt\\
&= (\partial_{\tilde{q}^{(k)}}L)\alpha^{(k-1)}|_{t_i}^{t_f} - ((\partial_{\tilde{q}^{(k)}}L)\alpha^{(k-2)}|_{t_i}^{t_f} - \int_{t_i}^{t_f}(d_t)^2(\partial_{\tilde{q}^{(k)}}L)\alpha^{(k-2)}dt)\\
&\dots\\
&= \sum_{i=1}^{k}(-1)^{i-1}(\partial_{\tilde{q}^{(k)}}L)\alpha^{(k-i)}|_{t_i}^{t_f} + (-1)^k\int_{t_i}^{t_f}(d_t)^k(\partial_{\tilde{q}^{(k)}}L)\alpha\ dt
\end{aligned}$$

And if we assume constraints for the higher order derivatives of $\alpha(t)$, similar to the first order case, with

$$\alpha^{(k)}|_{t_i}^{t_f} = 0 \text{ for all } k = 0\dots n-1,$$

then the summation term goes away, and we are left with

$$\begin{aligned}
d_\epsilon\int_{t_i}^{t_f} L(\tilde{q},\dot{\tilde{q}},\dots,\tilde{q}^{(n)},t)dt &= \int_{t_i}^{t_f}(\partial_{\tilde{q}}L)\alpha\ dt + \sum_{i=1}^{n}\int_{t_i}^{t_f}(\partial_{\tilde{q}^{(i)}}L)\alpha^{(i)}dt\\
&= \int_{t_i}^{t_f}(\partial_{\tilde{q}}L)\alpha\ dt + \sum_{i=1}^{n}(-1)^i\int_{t_i}^{t_f}(d_t)^i(\partial_{\tilde{q}^{(i)}}L)\alpha\ dt\\
&= \int_{t_i}^{t_f}((\partial_{\tilde{q}}L)+\sum_{i=1}^{n}(-1)^i(d_t)^i(\partial_{\tilde{q}^{(i)}}L))\alpha\ dt\\
&= 0 \text{ for all } \alpha \text{ at } \epsilon = 0.
\end{aligned}$$

This gives us the generalized form of the Euler-Lagrangian, with

$$(\partial_q L) + \sum_{i=1}^{n}(-1)^i(d_t)^i(\partial_{q^{(i)}}L) = 0$$

5. Brachistochrone Problem

Similar to the solution to Exercise 3, we can see that the shape of the curve $(x, y(x))$ should satisfy the equation

$$d_t(L-(\partial_{\dot{q}}L)\dot{q}) = 0 \text{ where } L = \frac{1}{\sqrt{2g}}\sqrt{\frac{1+(y')^2}{y}}.$$

From this we get

$$d_t(L-(\partial_{\dot{q}}L)\dot{q})) = \frac{1}{\sqrt{2g}}d_t(\sqrt{\frac{1+(y')^2}{y}} - \frac{(y')^2}{\sqrt{(1+(y')^2)y}}) = \frac{1}{\sqrt{2g}}d_t(\frac{1}{\sqrt{y(1+(y')^2)}}) = 0$$

Which leads to

$$y(1+(y')^2) = c^2 \text{ for some constant } c$$

With separation of variables on $y'^2 = \frac{c^2-y}{y}$, and the fact that we know $y' \leq 0$ due to the downwards motion of our particle as caused by the direction of gravity, we know that

$$\int -\sqrt{\frac{y}{c^2-y}}dy = x + d_1 \text{ for some constant } d_1$$

By evaluating the left hand side, first with a change of variables $u = \sqrt{c^2-y}$, $dy = -2\sqrt{c^2-y}\ du$

$$-\int \sqrt{\frac{y}{c^2-y}}dy = 2\int \sqrt{c^2-u^2}du$$

Then with a change of variables $v = \sin^{-1}(\frac{u}{c}) = \sin^{-1}(\frac{\sqrt{c^2-y}}{c})$, $du = c\cos v\ dv$, we get

$$-\int \sqrt{\frac{y}{c^2-y}}dy = 2\int \sqrt{c^2-u^2}du = 2c^2\int \cos^2 v\ dv = c^2(v+\sin v\cos v) + d_2 = x + d_1$$

Instead of substituting y back into this equation, we can directly see that

$$x = c^2(v+\sin v\cos v) + (d_2-d_1)$$
$$y = c^2\cos^2 v$$

Furthermore, by trigonometric identities of $\alpha = 2v$, we can get $\cos\alpha = 2\cos^2 v - 1$ and $\sin\alpha = 2\cos v\sin v$, leading to

$$x = \frac{c^2}{2}(\alpha+\sin\alpha) + (d_2-d_1)$$
$$y = \frac{c^2}{2}(1+\cos\alpha)$$

Rearrangement of the constants and variables $r = \frac{c^2}{2}$, $d = d_2 - d_1, \theta = \pi + \alpha$ gives us

$$x = r(\theta - \sin\theta) + d - r\pi$$
$$y = r(1-\cos\theta) - r\pi$$

While translating coordinates (so that the initial position is at the origin and not $(d-r\pi, -r\pi)$) gives us

$$x = r(\theta - \sin\theta)$$
$$y = r(1-\cos\theta)$$

6. Tautochrone Problem

For notation consistency, we will first suppose $r = -r$ from the previous problem.
From the laws of energy conservation in physics we get $\frac{1}{2}mv^2 + mgh = mgh_0$ in an ideal environment with zero initial velocity. Translated into this problem, we get

$$v^2 = 2g(h_0-h) = 2gr(\cos\theta - \cos\theta_i), \text{ and } v = \sqrt{2gr(\cos\theta-\cos\theta_i)}.$$

Now we know that the $v = \frac{dl}{dt}$, $dt = \frac{dl}{v}$ with l being distance traveled or arc length covered, which in this case would be

$$dl = \sqrt{(d_\theta x)^2 + (d_\theta y)^2}\ d\theta = \sqrt{r^2(1-\cos\theta)^2 + r^2\sin^2\theta}\ d\theta = \sqrt{r^2(2-2\cos\theta)}\ d\theta.$$

Upon integrating on both sides of $dt = \frac{dl}{v}$, this gives us

$$t = \int \frac{\sqrt{r^2(2-2\cos\theta)}}{\sqrt{2gr(\cos\theta-\cos\theta_i)}}d\theta = \int \sqrt{\frac{-r(1-\cos\theta)}{g(\cos\theta_i-\cos\theta)}}d\theta$$

Since we want to calculate the time T needed to move to the bottom of the curve, we see from $y = r(1-\cos\theta)$ with $r < 0$ that the minimal y within $\theta \in [0,\pi]$ occurs at π, so the limits of our integral should be expressed as

$$T = \int_{\theta_i}^{\pi} \sqrt{\frac{-r(1-\cos\theta)}{g(\cos\theta_i-\cos\theta)}}d\theta.$$

To evaluate this integral, we first use the half angle rule $\cos\theta = \sqrt{\frac{1+\cos 2\theta}{2}}$ and $\sin\theta = \sqrt{\frac{1-\cos 2\theta}{2}}$ to obtain

$$\begin{aligned}
T &= \int_{\theta_i}^{\pi} \sqrt{\frac{-r(1-\cos\theta)}{g(\cos\theta_i - \cos\theta)}} d\theta \\
&= \sqrt{\frac{-r}{g}} \int_{\theta_i}^{\pi} \sqrt{\frac{(1-\cos\theta)}{2} \frac{2}{(\cos\theta_i - \cos\theta)}} d\theta \\
&= \sqrt{\frac{-r}{g}} \int_{\theta_i}^{\pi} \sin\frac{\theta}{2} \sqrt{\frac{1}{(\cos^2\frac{\theta_i}{2} - \cos^2\frac{\theta}{2})}} d\theta
\end{aligned}$$

Substituting $\alpha = \frac{\theta}{2}$, $d\theta = 2d\alpha$ into the integral, we get

$$T = 2\sqrt{\frac{-r}{g}} \int_{\frac{\theta_i}{2}}^{\frac{\pi}{2}} \sin\alpha \sqrt{\frac{1}{(\cos^2\frac{\theta_i}{2}) - \cos^2\alpha}} d\alpha$$

And further substituting with $u = \frac{\cos\alpha}{\cos\frac{\theta_i}{2}}$, $d\alpha = -\frac{cos\frac{\theta_i}{2}}{\sin\alpha} du$, we get

$$\begin{aligned}
T &= 2\sqrt{\frac{-r}{g}} \int_{1}^{0} -\cos\frac{\theta_i}{2} \sqrt{\frac{1}{(\cos^2\frac{\theta_i}{2})(1-u^2)}} du \\
&= 2\sqrt{\frac{-r}{g}} \int_{0}^{1} \sqrt{\frac{1}{(1-u^2)}} du \\
&= 2\sqrt{\frac{-r}{g}} \sin^{-1} u \Big|_0^1 \\
&= 2\sqrt{\frac{-r}{g}} (\frac{\pi}{2} - 0) = \sqrt{\frac{-r}{g}}\pi
\end{aligned}$$

Which is

$$\sqrt{\frac{r}{g}}\pi$$

when we use r as defined in the previous problem.

We have thus shown that on a brachistochrone curve, the time it takes for a particle on the curve to reach the lowest point (via laws of motion) is independent of its starting point. Therefore, the brachistochrone curve is a type of tautochrone curve.