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Solutions of Exercise Sheet 2: Classical Variational Problems

Exercise 1 *(Beltrami Identity)*

The simplification is pretty straight forward and leads to

$$L - \dot{q}\frac{\partial L}{\partial \dot{q}} = \mathsf{const.}$$

Exercise 2 *(Shortest Path)*

The substitution of

$$L(y(x)) = \sqrt{1 + (y'(x))^2}$$

into the Euler-Lagrange equation (the Beltrami identity can be used for simplification) leads to $y''(x) = \mathsf{const.}$ which describes a straight line.

Exercise 3 *(Soap Film)*

The substitution of

$$L(y(x)) = 2\pi\, y(x)\sqrt{1 + (y'(x))^2}$$

into the Euler-Lagrange equation (the Beltrami identity can be used for simplification) leads to

$$y'(x) = \sqrt{\frac{y^2(x)}{c^2} - 1}$$

with constant $c \in \mathbb{R}$. Separation of variables leads to the solution

$$y(x) = c\cosh\left(\frac{x}{c} - \ln(c)\right).$$ [1]

Exercise 4 *(Higher Order Euler-Lagrange Equation)*

For $L = L(q, \dot{q}, \ldots, q^{(n)}, t)$ a derivation of an Euler-Lagrange equation analog can be derived similarly to the derivation in the lecture. The final equation is given by

$$\frac{\partial L}{\partial q} + \sum_{i=1}^{n}(-1)^i \frac{\mathrm{d}^i}{\mathrm{d}t^i}\frac{\partial L}{\partial q^{(i)}} = 0.$$

Exercise 5 *(Brachistochrone Problem)*

The substitution of

$$L(y(x)) = \frac{1}{\sqrt{2g}}\sqrt{\frac{1 + (y'(x))^2}{y}}$$

into the Euler-Lagrange equation (the Beltrami identity can be used for simplification) leads to

$$y'(x) = \sqrt{\frac{2r}{y(x)} - 1}$$

[1] This curve is also denoted with the term "catenary" derived from the Latin word for "chain", since it describes an idealized hanging chain under its own weight when supported only at its ends.

with $r = 1/(2c_1^2)$ and constant $c_1 \in \mathbb{R}$. Separation of variables and the application of trigonometric identities further lead to the parameterized representation

$$x(\varphi) = r(\varphi - \sin(\varphi)) + c_2, \quad y(\varphi) = r(1 - \cos(\varphi))$$

with additional constant $c_2 \in \mathbb{R}$.[2]

Exercise 6 *(Tautochrone Problem)*

The law of conservation of energy leads to the expression

$$T = \sqrt{\frac{r}{g}} \int_{\varphi_i}^{\pi} \sqrt{\frac{1 - \cos(\varphi)}{\cos(\varphi_i) - \cos(\varphi)}} \, \mathrm{d}\varphi$$

for the movement time of a particle placed initially at point $(x(\varphi_i), y(\varphi_i))$ with $0 < \varphi < \pi$ to the bottom. Using trigonometric identities and suitable substitutions we get

$$T = 2\sqrt{\frac{r}{g}} \int_0^1 \frac{1}{\sqrt{1 - u^2}} \, \mathrm{d}u = \sqrt{\frac{r}{g}}\pi.$$

[2]The resulting curve can be constructed by tracing a point on the rim of a wheel as it rolls smoothly along a straight line. It is denoted with the term "cycloid", which derives from the Greek "cyclo" meaning "circle" and "eidos" meaning "similar".