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High Fidelity Algorithmics Group
CS Dept. Stanford University

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CS448J: CASVC 2015 @ Stanford
Exercise Sheet 2: Classical Variational Problems

Exercise 1 *(Beltrami Identity, 4 Points)*

Derive a simplified version of the Euler-Lagrange equation under the assumption, that the Lagrangian is not explicitly dependent on time.[1]

Exercise 2 *(Shortest Path, 4 Points)*

Prove that the shortest path between two points on a plane is given by a straight line.

Exercise 3 *(Soap Film, 4 Points)*

Determine the shape of a soap film spanned between two parallel congruent circles. Assume, that the shape has a minimal surface and is rotational symmetric.
(Hint: The curved surface area of a solid figure obtained by rotating a curve y around the abscissa is given by the integral over $L(y(x)) = 2\pi\, y(x)\sqrt{1 + (y'(x))^2}$.)

Exercise 4 *(Higher Order Euler-Lagrange Equation, 6 Points)*

Derive an analog of the Euler-Lagrange equation for the more general case, in which the Lagrangian can also be dependent on derivatives of higher order.

Exercise 5 *(*Brachistochrone Problem, 6 Bonus Points)*

The historical origin of variational calculus is related to the Brachistochrone problem, which was solved by Johann Bernoulli in 1696. It asks for the shape of a curve down on which a particle accelerated by gravity will move in minimal time without friction.[2] Solve the brachistochrone problem.
(Hint: From the law of conservation of energy follows that the movement time can be calculated by integrating the Lagrangian

$$L(y(x)) = \frac{1}{\sqrt{2g}}\sqrt{\frac{1 + (y'(x))^2}{y}}$$

in which g denotes the gravitational constant.)

Exercise 6 *(*Tautochrone Problem, 6 Bonus Points)*

The historical tautochrone problem asks for a curve down on which a particle placed anywhere will move to the bottom in the same amount of time.[3,4] Prove that the solution of the Brachistochrone problem is a tautochrone curve.

[1] The resulting equation is called Beltrami identity, named after Eugenio Beltrami, who derived it in the 19th century.

[2] The term "Brachistochrone" derives from the Greek "brachistos" meaning "the shortest" and "chronos" meaning "time".

[3] This unique property of the solution curve of the Brachistochrone problem was first discovered by Christiaan Huygens in 1673 before it was known, that this curve has also the Brachistochrone property.

[4] The term "tautochrone" derives from the Greek "tauto" meaning "same" and "chronos" meaning "time".

Notes

- The Homework is due by 10:30am on Oct. 13. Written solutions should be handed in before the lecture. Programming assignments must be submitted by email to your tutor David Hyde <dabh@stanford.edu>.
- In case you have any questions about the assignments, please contact your tutor David Hyde <dabh@stanford.edu> or the instructor Prof. Dominik L. Michels <michels@cs.stanford.edu> directly via email.
- Office hours are every Friday, 10-12 in 208/209 Gates CS Bldg. or by appointment.
- The university expects both faculty and students to respect and follow Stanford's Honor Code; see https://communitystandards.stanford.edu/.