Exercise 5 (Analysis of Algorithms)

The loop terminates with a probability of

\[ P(i) = \left( \frac{n-1}{n} \right)^{i-1} \frac{1}{n} \]

in iteration \( i \). Hence the overall likelihood is given by

\[ \lim_{m \to \infty} \left( \sum_{i=1}^{m} P(i) \right) = \lim_{m \to \infty} \left( \sum_{i=0}^{m-1} P(i+1) \right) = \frac{1}{n} \sum_{i=0}^{\infty} \left( \frac{n-1}{n} \right)^{i} = 1, \]

in which we used the geometric series

\[ \sum_{i=0}^{m} q^i = \frac{1-q^{m+1}}{1-q} \quad \text{as} \quad m \to \infty \]

in the last step. It converges because of

\[ q = \left| \frac{n-1}{n} \right| < 1. \]

Therefore the loop terminates almost surely.

The average runtime is given by

\[ T(n) = \sum_{i=1}^{\infty} (iP(i)) = \frac{1}{n} \sum_{i=1}^{\infty} \left( i \left( \frac{n-1}{n} \right)^{i-1} \right). \]

Calculating the derivative of the geometric series on both sides leads to

\[ \frac{d}{dq} \left( \sum_{i=0}^{\infty} q^i \right) = \sum_{i=1}^{\infty} (iq^{i-1}) \]

and

\[ \frac{d}{dq} \left( \frac{1}{1-q} \right) = \frac{1}{(1-q)^2} \]

for \( |q| < 1 \) and therefore to the identity

\[ \sum_{i=1}^{\infty} (iq^{i-1}) \quad |q| \leq 1 = \frac{1}{(1-q)^2}. \]

From this we easily get the formula

\[ T(n) = \frac{1}{n} \cdot \frac{1}{(1-\frac{n-1}{n})^2} = n \]

for the average runtime. Hence the loop terminates after \( n \) steps in average.