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Solutions of Exercise Sheet 0: Analysis of Algorithms

Exercise 5 *(Analysis of Algorithms)*

The loop terminates with a probability of

$$P(i) = \left(\frac{n-1}{n}\right)^{i-1} \frac{1}{n}$$

in iteration i. Hence the overall likelihood is given by

$$\lim_{m\to\infty}\left(\sum_{i=1}^{m} P(i)\right) = \lim_{m\to\infty}\left(\sum_{i=0}^{m-1} P(i+1)\right) = \frac{1}{n}\sum_{i=0}^{\infty}\left(\frac{n-1}{n}\right)^{i} = 1,$$

in which we used the geometric series

$$\sum_{i=0}^{m} q^i = \frac{1-q^{m+1}}{1-q} \overset{m\to\infty}{\to} \frac{1}{1-q}$$

in the last step. It converges because of

$$q = \left|\frac{n-1}{n}\right| < 1.$$

Therefore the loop terminates almost surely.

The average runtime is given by

$$T(n) = \sum_{i=1}^{\infty}(iP(i)) = \frac{1}{n}\sum_{i=1}^{\infty}\left(i\left(\frac{n-1}{n}\right)^{i-1}\right).$$

Calculating the derivative of the geometric series on both sides leads to

$$\frac{d}{dq}\left(\sum_{i=0}^{\infty} q^i\right) = \sum_{i=1}^{\infty}\left(iq^{i-1}\right)$$

and

$$\frac{d}{dq}\left(\frac{1}{1-q}\right) = \frac{1}{(1-q)^2}$$

for $|q| < 1$ and therefore to the identity

$$\sum_{i=1}^{\infty}\left(iq^{i-1}\right) \overset{|q|<1}{=} \frac{1}{(1-q)^2}.$$

From this we easily get the formula

$$T(n) = \frac{1}{n}\frac{1}{\left(1-\frac{n-1}{n}\right)^2} = n$$

for the average runtime. Hence the loop terminates after n steps in average.